Engineering Science Series

DESCRIPTIVE GEOMETRY

ENGINEERING SCIENCE SERIES

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DESCRIPTIVE GEOMETRY

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PREFACE

This book represents a teaching experience of more than twenty years on the part of both of the authors at the Massachusetts Institute of Technology. In presenting such a book for the consideration of other institutions and of the public at large, it is well to note the relation of this course in descriptive geometry to the general study scheme, in order that the point of view of the authors may be better understood.

In any professional drawing of an engineering or architectural nature, especially if the element of design occurs, the draftsman or designer must see clearly the conditions in space. This has been well expressed by saying that he must be able to think in three dimensions. Such an ability is natural to but few. Fortunately, however, it is a power which can be acquired, more or less readily, by the great majority. This power can be gained, and it usually is by the so-called "practical" draftsman, by the simple process of making and remaking working drawings. But the authors believe that the same power can be acquired much more rapidly, and when acquired can be more forcefully and efficiently applied in designing, through the study of descriptive geometry.

The point of view in this text is therefore that of the draftsman. Mathematical formulæ and analytic computations have been almost entirely suppressed. It has been found that the students readily apply their knowledge of the theoretical mathematics to a finished drawing. For example, they make trigonometric computations from drawings with considerable facility. On the other hand, in applying even the simplest principles of solid geometry during the construction of a drawing the student is often anything but facile. The method of attack throughout this book is intended to be that which

shall most clearly present the actual conditions in space. Wherever experience has shown that a simple plan and elevation are not amply sufficient for this purpose, additional views or projections have been introduced freely, corresponding to the actual drafting practice of making as many side views or cross sections as may be needed.

In the matter of arranging the views in a practical drawing, there has been, still is, and probably always will be, controversy. Architects, who must embellish their drawings with shades and shadows, as well as some engineers, prefer to place the plan below the elevation, according to the so-called first quadrant projection. Entire text-books, noticeably the older ones, have been prepared with all the objects and all the theory studied in the first quadrant. Other engineers, mechanical especially, insist that the only proper placing of the views is to have the plan above the elevation, according to the method of the third quadrant. In consequence, some recent text-books have been issued with all the work in the third quadrant. The authors have no desire to take sides in this controversy, but believe in giving the student practice in both of these methods. Hence some of the problems are studied in the first quadrant, others in the third. It has been found that the student has no greater difficulty in working in one quadrant than in the other, while considerable additional power is gained, and a clearer insight into space in general is obtained, by not restricting the work to a particular corner of space.

A requirement of practical drafting is that the construction shall be confined to the limits of the drawing board. There is always a limit beyond which points cannot be made available. A similar requirement, namely, that all the points used in the construction must be within the limits of the figure, is rigidly insisted upon in the authors' classes. This frequently involves an auxiliary construction not embodied in the general theory of the problem. Numerous auxiliary constructions of this kind are given throughout the book. The additional grasp of the subject obtained by being obliged thus to force

his way through whatever difficulties may arise, has been found of great value to the student in his subsequent work.

The notation adopted in this book is a modification of that first suggested and used by Professor William Watson of the Institute of Technology (around 1870). The modifications have been made from time to time by the authors in the course of their teaching experience. The authors have examined many books; and when opportunity offered, they have questioned students coming from other institutions in their search for a clear system of notation. While the system here given is not above criticism, the authors are firmly of the opinion that it is the best that has ever come to their attention.

Although this book has been prepared with the needs of one particular institution in mind, it is hoped that it will not be found wanting in general interest.

ERVIN KENISON,
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
September, 1917.

PREFACE TO THE ENLARGED EDITION

THE principal changes in the present edition are additions of new material, namely, a chapter on Warped Surfaces, and a number of Examples for Solution.

The original book was prepared with the needs of but one institution in mind. In consequence, certain topics, usually given in elementary texts, were omitted. This omission has been corrected. The book now covers with, it is hoped, a satisfactory degree of completeness, the ground usually included in elementary treatises on descriptive geometry. From this text may be laid out for beginners in the subject a course of from two hundred to two hundred and fifty school hours — and the material can, it is felt, be adapted to the requirements in any engineering or technical school.

The examples for solution have been prepared in response to numerous demands, and should materially assist the teacher who, through lack of time or experience, is unable to prepare his own. There are at least three to five times as many as can be solved by any individual student, thus allowing for considerable variation in the course as given from year to year.

As a result of the enlargement of this book, it is believed that its usefulness is increased.

E. K.
H. C. B.

SEPTEMBER, 1923.

CONTENTS

DESCRIPTIVE GEOMETRY

DESCRIPTIVE GEOMETRY

INTRODUCTION

1. Descriptive Geometry. Descriptive Geometry is an applied science which treats of the graphical representation of lines, planes, surfaces, and solids, and of the solution of problems concerning size and relative proportions. Thus it lies at the foundation of all architectural and mechanical drafting.

While the study of descriptive geometry does not require extended mathematical knowledge, and while its operations are not strictly mathematical, the best results cannot be obtained without some acquaintance with the principles of plane and solid geometry.

Descriptive geometry differs from analytic geometry of three dimensions in that the solutions are based, not on algebraic equations, but on drawings. In a certain sense, this greatly increases the scope of descriptive geometry. By the graphical processes, objects of any form whatever, no matter how irregular, or whether any equations for them exist or not, may be treated. On the other hand, descriptive geometry resembles analytic geometry in that the drawings, like the equations, are merely *representations* of the conditions in space.

By the methods of descriptive geometry the solution of any problem involving three dimensions consists of three distinct processes, as follows:

(1) Representation of the lines, planes, surfaces, or solids in space by corresponding plane figures.

(2) Solution of the problem by the use of the plane figures.

(3) Determination of the relation in space which corresponds to this solution.

In order that these processes may lead to a successful result, it is evident that it must be possible to pass without ambiguity from the object in space to its representations, and also without ambiguity from the representations to the object in space again.

2. Visualization. The process of passing from the representations to the object in space is a purely mental one. It is, therefore, apt to be ignored by the student at the beginning of his course, to his detriment later on. The process is called "visualizing" or "reading" the drawing, and is absolutely essential in any practical application of descriptive geometry. (See § 3.) Because of its importance, both in theory and in practice, considerable emphasis will be laid on visualization in the present text. The student is advised to accustom himself early to look upon his drawings as *representations* of conditions in space, and to work out his problem by considering the space relations involved, rather than by the (plane) geometrical relations existing in the drawing itself.

3. Practical Application of Descriptive Geometry. In the design of all engineering and architectural structures, as machines, bridges, buildings, etc., as well as in many of the less pretentious mechanic arts, there comes a time when the forms and arrangements of the various parts must be considered. The problem then becomes one which is solved, either wholly or in part, by the graphical methods of descriptive geometry. Moreover, the drawings, once made, take the place of written description, and form the *language* by means of which the designer conveys his ideas to the builder or mechanic, who, after "reading" (visualizing) them, can build or construct the work.

In addition to the usefulness of descriptive geometry in its practical applications, it is the conviction of the authors that its fundamental and educational value lies in its unique power to develop the mental concept, or visualization, the importance of which has already been emphasized.

CHAPTER I

ELEMENTARY PRINCIPLES — THE COÖRDINATE PLANES

4. Orthographic Projection or Orthographic View. The terms **orthographic projection** and **orthographic view**, or more simply **projection** and **view**, are used to denote a drawing which represents, in accordance with a certain artificial and conventional system of vision hereafter described, some object, surface, line, point, or combination of these, that has a definite position in space. Such a drawing may be made on any plane surface, as, for example, a sheet of paper or a blackboard.

5. Projection on a Single Plane. In Fig. 1, let Q represent any horizontal plane in space, and A a square right prism with

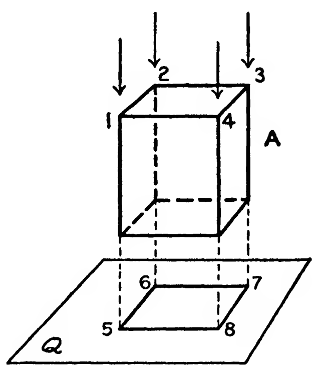

FIG. 1.

its bases parallel to the plane. If the prism be looked at from above, at right angles to the plane, the arrows representing the lines of sight, the square top, 1–2–3–4, will be the only part seen. If this square be now imagined to drop vertically until it lies in the plane Q in the position 5–6–7–8, the latter would be the **orthographic projection** of the prism on the plane Q. The plane Q is called a **plane of projection.** In this case the projection is a top view of the given object.

In looking at any object naturally, it is a familiar fact that the lines of sight all converge to the eye. In orthographic projection, however, this is not the case; instead, the lines of sight are all assumed to be parallel, and every view or projection is made on this basis. On account of this difference, the resulting views are more or less unlike those seen with the natural eye. With natural vision, the same object, placed at varying distances from the eye, appears smaller when farther away, while in orthographic projection the size of the view or projection is not affected by the distance of the object from the observer.

Figures 2–10 show pictorially the projections of objects of simple form. Figures 2, 3, 4, 5, 6, and 9 are projections on a

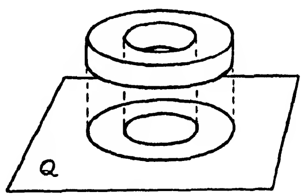

FIG. 2.

horizontal plane, and are top views. Figures 7, 8, and 10 are projections on a vertical plane, and are front views. In each case the plane Q is a plane of projection.

From these figures it is evident that one view of an object is not sufficient to determine its size and shape, hence two or more views or projections, requiring as many planes of projection, must be used.

6. Choice of the Planes of Projection; the Coördinate Planes. The simplest and most advantageous angle between the planes is a right angle.

At any given point on the earth's surface, there are two natural mutually perpendicular directions, which must always be considered in the applied arts and sciences: the *horizontal* or level, as shown by the surface of still water, and the *vertical* or plumb, as shown by a freely falling body.

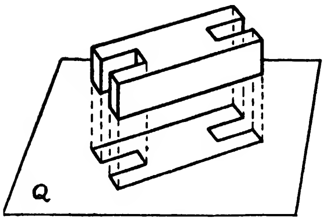

FIG. 3.

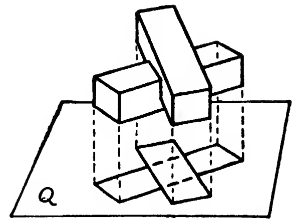

FIG. 4.

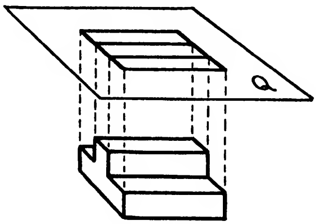

FIG. 5.

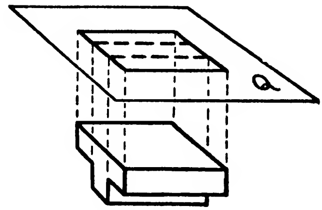

FIG. 6.

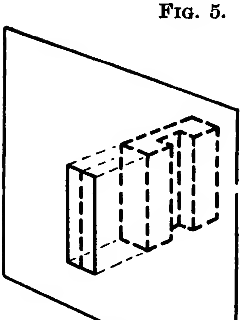

FIG. 7.

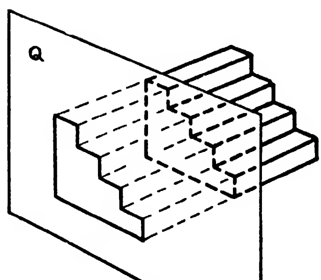

FIG. 8.

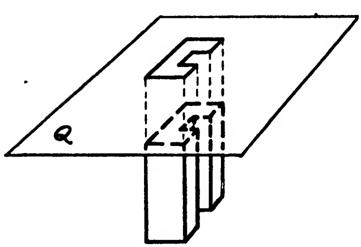

FIG. 9.

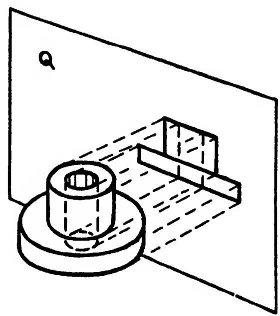

FIG. 10.

The planes of projection are chosen, accordingly, the first horizontal, the second perpendicular to the first, and therefore vertical. These two planes are known as the **horizontal** and **vertical coördinate planes,** respectively, and will be designated by the letters *H* and *V*.

At any given place the horizontal plane is always fixed in direction, since all horizontal planes are parallel. But any plane perpendicular to the horizontal is vertical. Hence the direction of the vertical plane is to some extent arbitrary.

7. Names of the Views. A view or orthographic projection made on a horizontal plane of projection, *H*, is known as a **horizontal projection,** usually abbreviated to ***H*-projection.** This term is equivalent to the expression *top view* which was used above, and which may still be employed. In practice, however, it is usually called the **plan.** Thus, the terms *top view, plan,* and *horizontal projection* are synonymous. Similarly, a view made on a vertical plane of projection, *V*, is called a **vertical projection,** or a ***V*-projection,** or a **front view,** or an **elevation.**

8. Position of the Observer with Reference to the Coördinate Planes. When making a vertical projection or front view, the observer is in a position squarely facing the *V*-plane, and either above or below *H*, as the size and position of the object may require. When making a top view or *H*-projection, the position is as if the observer were to stand on or above the *H*-plane, facing the *V*-plane, then to bend forward and look vertically down upon the given object and the horizontal plane. Except in rare instances, not considered here, the *V*-plane is never viewed from the back or further side, nor the *H*-plane from underneath.

9. The Ground Line and the Four Quadrants. Let the coördinate planes be chosen as in Figs. 11 and 12. The line of intersection, *GL*, of these planes is known as the **ground line** — a somewhat unfortunate name, since it is derived from but one of the properties of the line, and ignores another equally important property, as will be seen later. The ground line divides each of the coördinate planes into two parts.

Although a pictorial representation can show only a limited portion of each plane, the coördinate planes are supposed indefinite in extent. Hence they divide the whole of space into four **quadrants**, numbered I, II, III, and IV, as follows:

QUADRANT I: above *H* and in front of (or on the near side of) *V*.

QUADRANT II: above *H* and behind (or on the far side of) *V*.

QUADRANT III: below *H* and behind *V*.

QUADRANT IV: below *H* and in front of *V*.

10. The Relation between an Object and Its Projection. In Fig. 1, the projection 5–6–7–8 of the top of the prism might be obtained by extending the vertical edges of the prism until they intersect the plane of projection, *Q*. The point 5 is the projection of point 1, 6 of 2, and so on. In general, the (orthographic) projection of a point on any plane may be defined as the foot of a perpendicular from the point to the plane.

The projection of any object is composed of the projection of all its points, found in the same manner.

11. The Relation between Two Projections of an Object. Let a rectangular card, *abcd*, be placed as shown in Fig. 11. The card is in the first quadrant, above *H*, in front of and parallel to *V*. The long edges are vertical, the short edges parallel to *H*. The projection of the card on *V* is the equal rectangle $a^vb^vc^vd^v$, and on *H* the straight line $a^hb^hc^hd^h$, equal in length to the short edge. The perpendiculars dd^v and dd^h determine a plane which is perpendicular to both *H* and *V*, and therefore to their line of intersection, *GL*.

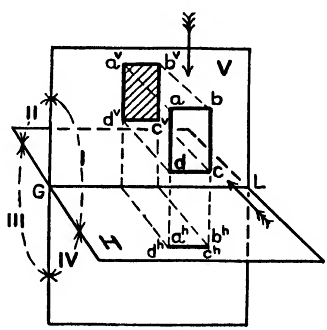

FIG. 11.

The same is true for the two perpendiculars from every other corner of the card. Hence *the two projections of any point, together with the point itself, must always be in the same plane*

perpendicular to GL. This is one of the fundamental relations of orthographic projection.

12. Notation. In this book, projections on the horizontal coördinate plane, H, will be denoted by the small letter, h, attached as an exponent to the letter or character which denotes the actual point, line, or object projected. Projections on the vertical coördinate plane, V, will be similarly denoted by the

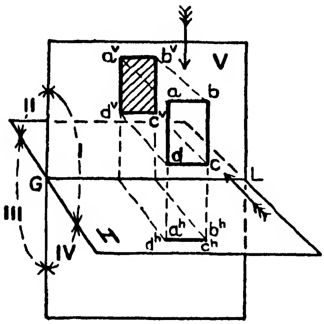

FIG. 11 (repeated).

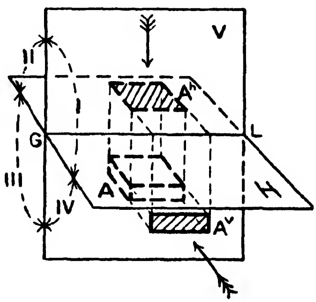

FIG. 12.

use of a small letter, v, as an exponent. Thus, in Fig. 11, point a is one corner of the card in space; the projection of a on H is lettered a^h; of a on V is called a^v. In Fig. 12, the object, A, a rectangular block placed in the third quadrant, is projected on H as A^h, on V as A^v. Further statements in regard to the notation will be made from time to time, when necessary.

13. Monge's Method. The use of two mutually perpendicular coördinate planes, as in Figs. 11 and 12, enables us to represent objects of three dimensions by plane figures. It is sometimes called Monge's method, after the originator. Drawings, however, are not made ordinarily on two drawing surfaces at right angles to each other. A single drawing surface (paper, cloth, board, etc.) is taken to represent both H and V.

14. Projections. In the pictorial representations, Figs. 1–12, the planes of projections have been shown limited in extent, in order to give a clear idea of their position. Actually, the

planes are unlimited in extent (§ 9), so that in making projections no outlines for the coördinate planes are shown.

The projections or views of the card of Fig. 11 are given in Fig. 13. When looking toward V, the horizontal plane will be

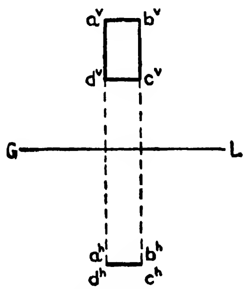

Fig. 13.

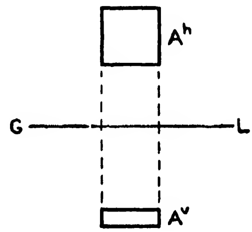

Fig. 14.

seen edgewise, and is represented by GL; the V-projection of the card will be the rectangle $a^vb^vc^vd^v$, as in Fig. 11. Looking down on the H-plane, the V-plane will be seen edgewise, and is usually represented by the same line, GL, previously used as the edge view of H. The card will also be seen edgewise, and will project on H in the straight line $a^hd^hb^hc^h$.

The projections, plan and elevation, of the rectangular block of Fig. 12 are given in Fig. 14 at A^h and A^v respectively.

Figure 15 shows the projections of the object of Fig. 3. Figure 16 shows the projections of the hollow ring of Fig. 2.

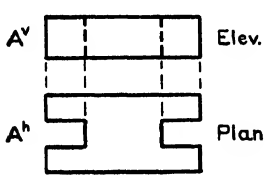

Fig. 15.

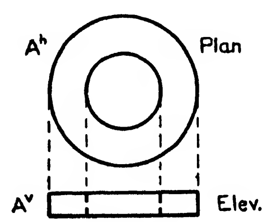

Fig. 16.

Since no ground line is given in Figs. 15 and 16, the quadrants in which these objects are placed are not known. However, this does not affect the size or the shape of the projections (§ 5).

CHAPTER II

PROJECTIONS OF THE POINT AND OF SIMPLE SOLIDS

15. Notation of the Point. An isolated point in space will be denoted, in general, by a small letter. A point lying in a line, or in the boundary of a surface or solid, will be denoted either by a small letter or by a number, according to convenience. The projection of any point on H or V will be designated by the suitable use of the small letters h and v as exponents (§ 12).

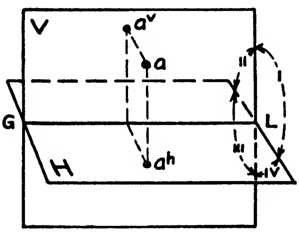

FIG. 17.

16. Projecting Isolated Points. Let a, Fig. 17, be a point situated in Quadrant I. By inspection of the figure, it is seen that this point projects on H in front of GL, and on V above GL. Let the points b, c, d, be placed in Quadrants II, III, IV, respectively. Visualizing in a similar manner, and using the notation of § 15, we may make the following table:

QUADRANT I:	a^h lies in H in front of GL.
	a^v lies in V above GL.
QUADRANT II:	b^h lies in H behind GL.
	b^v lies in V above GL.
QUADRANT III:	c^h lies in H behind GL.
	c^v lies in V below GL.
QUADRANT IV:	d^h lies in H in front of GL.
	d^v lies in V below GL.

By the method of the single ground line, Fig. 13, the drawing is divided into two parts by the line GL. The upper part represents both V above GL and H behind GL, the lower part represents both H in front of GL and V below GL. In this method, the lines joining a^h and a^v, b^h and b^v, etc., must be perpendicular to GL. The resulting projections are given in Fig. 18.

17. Projectors. The line aa^v or bb^v, Fig. 11, which projects the point on the plane of projection, is called a **projector.** A line like $a^v a^h$, Fig. 13 or Fig. 18, which connects the two projections of a point in the drawing, also is called a projector. These two uses of the word rarely cause confusion; if necessary

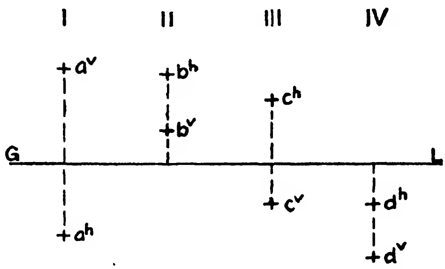

FIG. 18.

to distinguish, however, the terms *space projector*, in space, and *ruled projector*, in the drawing, may be employed.

18. Visualizing the Quadrants. The propositions of § 16 can be reversed, and the quadrant in space in which any point lies can be told at once from its projections. To do this merely by memorizing the relations of the two projections to the ground line is not visualizing in any sense, and is of little or no value.

As the first step in visualizing, let us inquire *which side* of each of the coördinate planes is represented by the drawing surface. Regardless of the position of the point or object represented, this is always the side of the plane which is nearer to an observer placed in the *first quadrant.* The same idea is expressed by saying that the *H*-plane is always viewed from above, and the *V*-plane from in front (§ 8).

Since the point may be on either side of the coördinate plane while the drawing is always viewed from the same side, it follows that it may be necessary to visualize the point in either of two ways: (1) nearer to the observer than its projection, that is, above or in front of the drawing; (2) beyond its projection, that is, below or behind the drawing and seen as if looking through a transparent surface.

19. Uses of the Quadrants. In Figs. 19–22 an isolated point is shown placed, successively, in the four quadrants. The remarks and deductions apply equally well to any object lying wholly within the quadrant considered.

First Quadrant (Fig. 19). The point is on the nearer side of the drawing in each projection. This quadrant is much

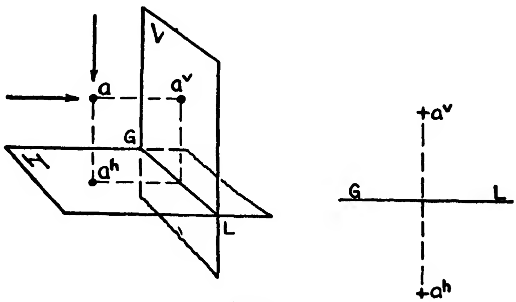

Fig. 19.

used in architectural work and occasionally in engineering work. On account of the comparative ease with which aids to visualization may be placed over, rather than under, the drawing, this quadrant is very generally used in the first presentation of a problem in descriptive geometry.

Second Quadrant (Fig. 20). The point is on the nearer

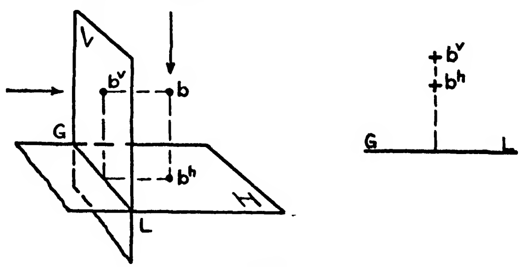

Fig. 20.

side of the *H*-projection, but on the farther side of the *V*-projection. The principal use of this quadrant is in the subject of perspective.

Third Quadrant (Fig. 21). The point lies beyond the plane of the drawing in each projection, and must be imagined

as if seen through transparent planes. This quadrant is used more frequently in practical drafting than any of the others.

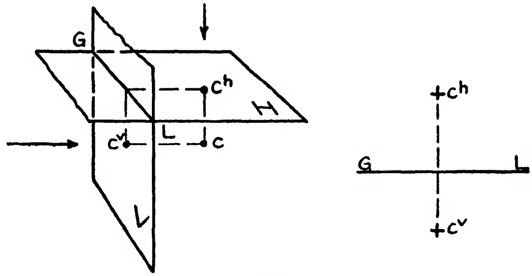

FIG. 21.

FOURTH QUADRANT (Fig. 22). This is analogous to the second quadrant, with *H* and *V* reversed. It is little used in practice.

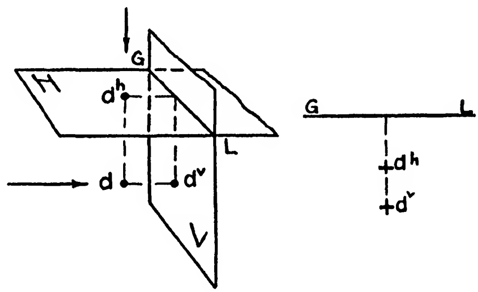

FIG. 22.

20. Views of a Prism. The plan and elevation of a hexagonal right prism placed in the third quadrant are given in Fig. 23. The prism is in a vertical position, its upper base in the *H*-plane. Visualize the solid in space. The distance *y*, the length of the prism, is the height of one end above the other. The distance *x* is the distance between the parallel sides, or the thickness of the prism.

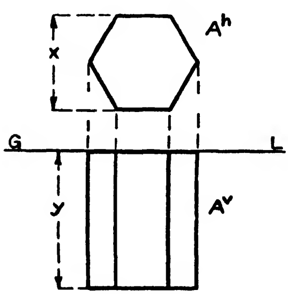

FIG. 23.

This illustrates one of the fundamental principles of orthographic projection: *an elevation, or V-projection, shows heights, or distances up and down ; a plan, or H-projection, shows distances from front to back.*

21. Distances of a Point from H and V. Let it be required to visualize the point m, Fig. 24. First, suppose the drawing paper to represent H; then m^v is ignored, while GL becomes the H-projection of the V coördinate plane. Hence m^h shows that the point m in space is not only in front of V, but also at the distance x from V. Now suppose the drawing paper to represent V; then m^h is ignored, GL becomes the V-projection of the H coördinate plane, and m^v shows that the point m is at the distance y below H.

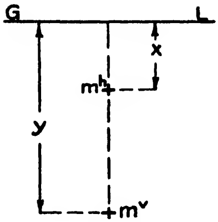

FIG. 24.

Conversely, if the distances of the point in space from H and V are given, the projections can be located at the given distances from GL. Which projection is determined by which distance?

22. Special Positions of the Point. In Fig. 25,
the point e lies in H, and in front of V;
the point f lies in H, and behind V;

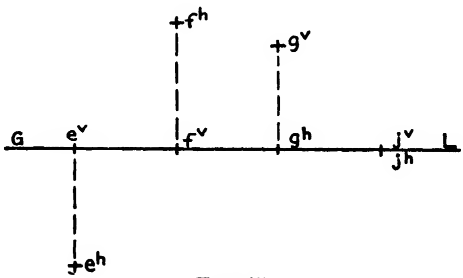

FIG. 25.

the point g lies in V and above H;
the point j lies in both H and V, that is, in the ground line.
Visualize each of these points.

23. Projections of Simple Solids. Solids bounded by plane faces, such as prisms, pyramids, wedges, frustums, etc., are projected by drawing the projections of the straight lines which form the edges of the solid. Only simple positions of these solids can be considered at present.

24. Visibility of Solid Objects. In viewing a solid, no matter from what point of view, only a portion of its surface is visible, while the rest is invisible. In any projection of a solid, therefore, there are usually both full and dotted lines, which represent respectively visible and invisible edges. The correct representation of these visible and invisible edges is an essential part of the projection of any solid. The student's facility in determining visibility is, to a considerable extent, a measure of his understanding of the problem.

In visualizing any object or objects, an *H*-projection, or plan, is always viewed from above; a *V*-projection, or elevation, from in front (see § 8). This is true, whatever be the relative positions of these projections with respect to the ground line, or to each other.

In a drafting office, the relative position of plan and elevation is usually prescribed by the custom of the office, so that it is known by the position on the sheet which view is to be read as an *H*-projection, and which as a *V*-projection. In studying the theory, however, this is not the case; objects may be placed in any position in any quadrant, and some indication as to which is the *H*-projection and which the *V*-projection must always be given. For the present, we shall do this by the notation, using the index letters ^{h} and ^{v} (§ 12).

25. Illustrative Examples. The student should visualize the solids projected in Figs. 26–32, aided by the following hints. Notice carefully the visibility in each case, as shown by the full and dotted lines (§ 24).

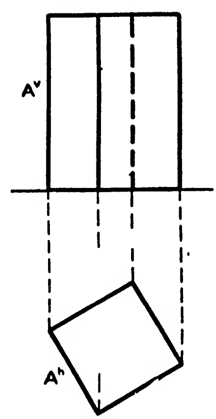

FIG. 26.

Figure 26. *First quadrant. A square right prism.* The base is in *H*, and the lateral edges are vertical. The *H*-projection shows the true size of the base, the distance of each lateral edge from *V*, and the angle between each lateral face and *V*. The *V*-projection shows the altitude of the prism.

Figure 27. *Third quadrant. A rectangular right prism.* The base is parallel to *H*, and the lateral edges are vertical. The *H*-projection shows the true size of the base, and its position (angles, distances) with respect to *V*. The *V*-projection shows the altitude of the prism and its distance from *H*.

Figure 28. *Fourth quadrant. A triangular right prism.* The base is parallel to *V*, and the lateral edges are perpendicular to *V*. The *V*-projection shows the true size and shape of the base, and its position with respect to *H*. Note that the base is not a regular (equilateral) triangle. The *H*-projection shows the length of the prism and its distance from *V*.

Figure 29. *Second quadrant. A square right pyramid.* The base is parallel to *H*. The *H*-projection shows its size and position. The *V*-projection shows the distance of the base from *H*, and the altitude of the pyramid.

Figure 30. *Third quadrant. A pentagonal right pyramid.* The base is parallel to *H*. The apex is below the base, so that the pyramid is inverted. The base is a regular pentagon. One edge of the base is perpendicular to *V*, so that one of the lateral faces projects on *V* as a straight line.

Figure 31. *First quadrant. A frustum of a regular hexagonal pyramid.* The lower base of the frustum is in *H*, the upper base is parallel to *H*. In order to aid in the visualization of the frustum, the complete pyramid is shown in light dotted lines.

Figure 32. *Quadrant indeterminate. An irregular triangular pyramid.* This solid may be constructed by assuming any four points in space, and then connecting each point with the other three, thus forming a solid bounded by four triangular faces. In visualizing this pyramid, any one of the four points may be taken as the vertex, and the other three points as the corners of the base.

In Fig. 32, no ground line is shown; hence the distances of the object from *H* and *V* cannot be told. Nevertheless, the projections represent a pyramid of definite size and shape (compare § 14, Figs. 15 and 16).

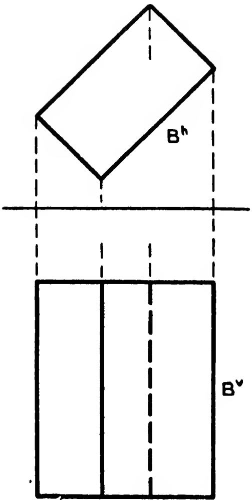

FIG. 27.

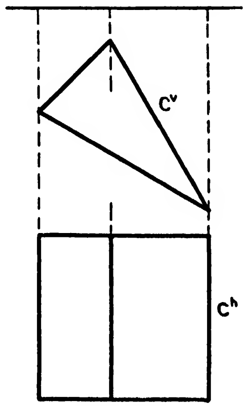

FIG. 28.

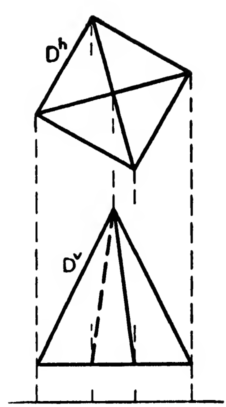

FIG. 29.

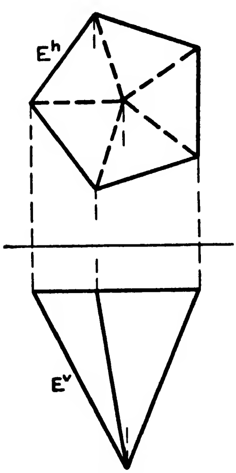

FIG. 30.

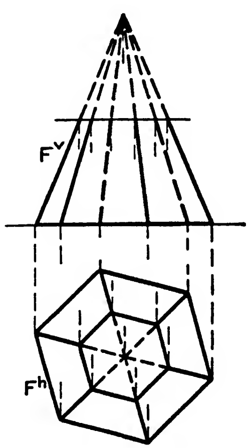

FIG. 31.

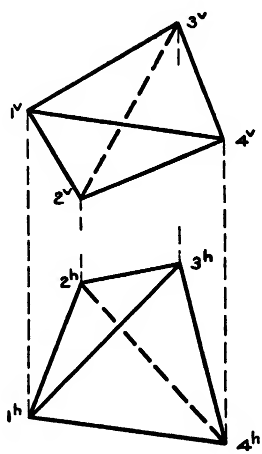

FIG. 32.

26. Curved Solids. In projecting solids bounded partly or wholly by curved surfaces, the projection cannot always be made wholly of edges of the solid, but may consist partly or wholly of the apparent outline or *contour* of the object.

A few typical curved solids are represented in Figs. 33–38.

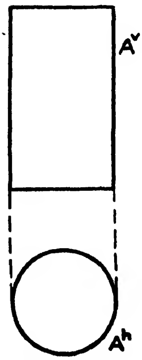

FIG. 33.

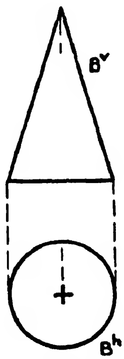

FIG. 34.

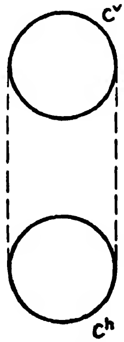

FIG. 35.

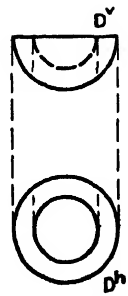

FIG. 36.

No ground line is shown, but the objects are all supposed to be in the first quadrant:

A — A circular right cylinder.

B — A circular right cone.

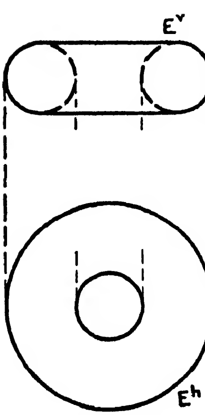

FIG. 37.

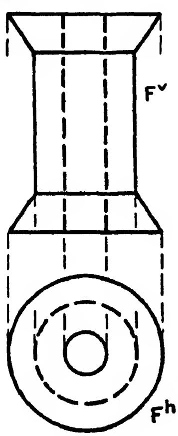

FIG. 38.

C — A sphere.

D — A hollow hemisphere, or hemispherical bowl.

E — A torus. This is the solid generated by revolving a circle about an axis situated in its own plane, but outside the circle.

F — A spool, consisting of a cylindrical body, conical ends, and with a cylindrical hole lengthwise through it.

CHAPTER III

REPRESENTATION OF THE STRAIGHT LINE — TRACES

27. The Straight Line. A line is the path of a moving point, and is not necessarily straight. Yet in ordinary use, the term *line*, by itself, and without anything to imply the contrary, always means a straight line.

28. Projections of the Straight Line (Fig. 39). Let cd be any straight line, and Q any plane of projection. Then as a point

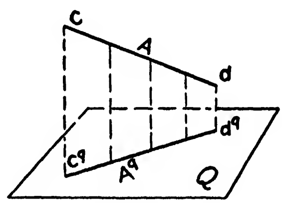

FIG. 39.

moves along the line from c to d, its projection will move along the plane from c^q to d^q. The following propositions are evident:

1. *The projection of a straight line is a straight line.*

2. *The projection of any point in the line lies in the projection of the line.*

Since any point in space is definitely determined when its projections on H and V are known, it follows that, in general, any two straight lines assumed at random, one in H and one in V, are the projections of one and only one straight line in space. Certain exceptions will be noted in § 34.

29. Notation of the Straight Line. A straight line of definite length will be given by the two points at its extremities, as the line cd ($c^h d^h$, $c^v d^v$). A straight line of indefinite length will be denoted by a single capital letter. This letter alone denotes the line in space, and the letter with a suitable index indicates a projection. Thus we shall say the line A, or the line $B(B^h, B^v)$

30. Visualizing the Straight Line. The following example illustrates a general method of visualization. A single projection, as we have seen, is not sufficient to locate a point or line in space. By a constant comparison of one projection with the other, however, a sufficient number of facts are brought out to complete the mental picture of the conditions in space.

Consider the line A, Fig. 40. In visualizing or "seeing" the line in space, either the H-projection or the V-projection may be used first as a basis. As the natural position of the drawing surface is horizontal, let the H-projection be chosen first. Then in viewing the line A from above, the line in space is either directly over or directly under A^h, or else partly above and partly below. Visualizing in this direction alone, however, can give no idea whether the line is parallel or oblique to the H-plane. By now looking toward V, and seeing the line in space projecting as A^v, the fact is revealed that the left-hand end of the line is the higher, and that, reading from this end, the direction or slope of the line in space is downward to the right. Returning to the H-projection, and reading along the line in the same direction, namely, from left to right, the line is seen to slope away from the observer, that is, backward. Finally, by comparison of the two projections, we find that the left-hand end of the line is in the first quadrant, the right-hand end in the third, the central portion in the second quadrant (§§ 16, 19).

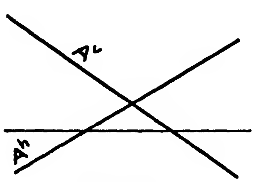

FIG. 40.

31. The Slope of a Line. In determining the slope of a line, the directions up and down, right and left, both of which are shown by the V-projection, rarely trouble the student. The directions forward and backward, however, which are always shown by the H-projection, often cause confusion. This difficulty should disappear by considering the method of projecting a solid object; for in projecting any actual object, the side nearest the observer is always considered to be the front of the object (see §§ 20, 24). A point which is farther away from

the observer is behind one which is nearer; so that the direction from front to back is *away from the observer.* This is shown in Fig. 41 by the arrow *A*, which points away from the observer, and therefore points backward. The arrow *B* in this figure points toward the observer, and is pointing forward.

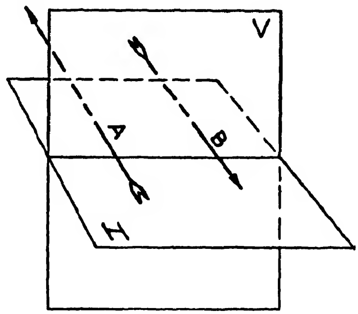

FIG. 41.

Points in the first and fourth quadrants are always in front of points in the second and third quadrants; but points on the same side of the *V*-plane must be read by the relative positions of their *H*-projections, the lower side of an *H*-projection being always the front.

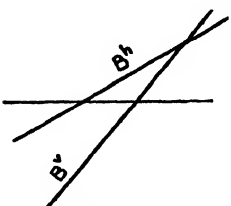
FIG. 42.

The slope of the line *A*, Fig. 40, would be described as downward, backward, to the right. The slope may also be given as upward, forward, to the left, since it is immaterial in which direction the line is read.

32. Additional Examples. Line *B*, Fig. 42. Read in the manner already described, it will be found that this line is oblique to *H* and *V*, passes through the fourth, third, and second quadrants, and slopes upward, backward, to the right (or downward, forward, to the left).

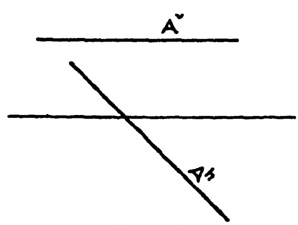
FIG. 43.

Line *A*, Fig. 43. The *V*-projection is parallel to the ground line, so that every point of the line is the same distance from *H*. The line *A* is *parallel* to *H*, oblique to *V*, passes through the second and first quadrants only, and slopes forward to the right (backward to the left).

NOTE. Lines parallel to *H* are known as ***H*-parallels**; lines parallel to *V* are known as ***V*-parallels**. A line parallel to *H* may also be called a horizontal line; but the corresponding term, *vertical line*, is limited to a line perpendicular to *H*. *H*- and *V*-parallels are very important lines.

33. Special Positions of the Line. The student should satisfy himself of the truth of the following propositions by direct visualization.

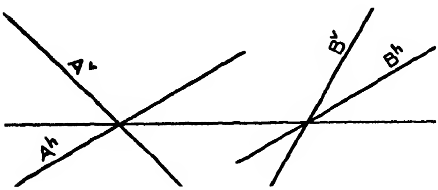

Fig. 44.

1. *If a line A intersects the ground line, A^h and A^v intersect on GL, and in the same point as A* (Fig. 44).

2. *Conversely, if A^h and A^v intersect the ground line in the same point, the line A intersects the ground line.* Through how many quadrants does such a line pass?

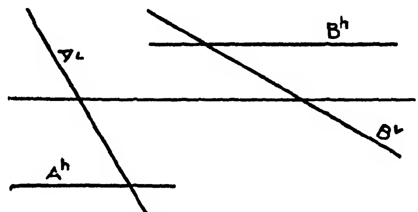

Fig. 45.

3. *If a line A is parallel to H, A^v is parallel to the ground line; and conversely, if A^v is parallel to the ground line, A is parallel to H* (Fig. 43).

4. *If A is parallel to V, A^h is parallel to the ground line; and conversely, if A^h is parallel to the ground line, A is parallel to V* (Fig. 45).

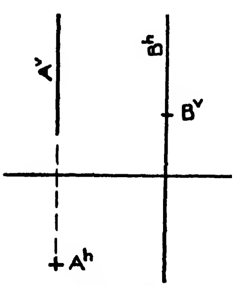

Fig. 46.

5. *If A is perpendicular to H, A^v is perpendicular to the ground line, and A^h is a point; conversely, if A^h is a point, A is perpendicular to H.* (See line A, Fig. 46.)

6. *If A is perpendicular to V, A^h is perpendicular to the ground line, and A^v is a point; conversely, if A^v is a point, A is perpendicular to V.* (See line B, Fig. 46.)

7. *If A lies in H, A^h coincides with A, while A^v coincides with the ground line* (Fig. 47). The statements of the analogous and converse propositions are left to the student.

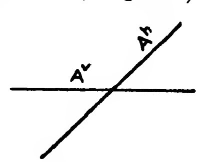

Fig. 47.

34. Exceptional Cases. While in general (§ 28) the two projections of a line may be assumed in any direction at random, certain exceptions are illustrated by the following examples :

(*a*) (Fig. 48). The projection A^h is perpendicular to the ground line, while the projection A^v is not. Assume on A^h any point c^h, and find c^v by projecting to A^v. The projector c^hc^v coincides with A^h, hence c^v is always the same point on A^v, no matter where c^h is chosen. Now let d^v be any point on A^v except c^v. The projector from d^v is parallel to A^h, and there

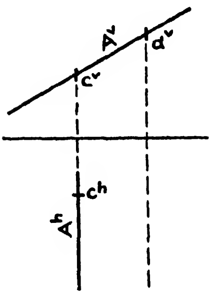

Fig. 48.

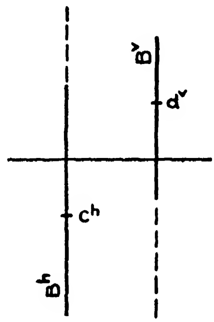

Fig. 49.

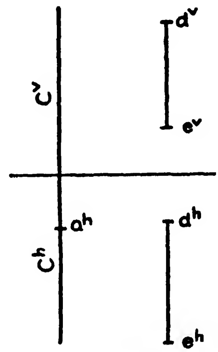

Fig. 50.

is no point d^h to correspond with d^v. Hence there is no line in space to correspond with the given projections.

An analogous case arises if A^v is taken perpendicular to the ground line, while A^h is not.

(*b*) (Fig. 49). The projections, B^h and B^v, are both perpendicular to the ground line, but at different points. Let c^h be any point on B^h; the projector from c^h is parallel to B^v, and there is no point on B^v to correspond with c^h. Similarly, if d^v is any point on B^v, the projector from d^v is parallel to B^h, and there is no point d^h. Therefore there is in space no line B corresponding to these two projections.

(*c*) (Fig. 50). The projections C^h and C^v are both perpendicular to the ground line, and at the same point. Let a^h be any point on C^h. The projector from a^h coincides with C^v, and the particular point, a^v, on C^v, which corresponds with a^h cannot be determined. The line C is therefore indeterminate. But if the line be projected by means of two of its points (§ 29), it at once becomes determined. Thus the line *de*, Fig. 50, is a definite line, lying in a plane perpendicular to the ground line.

35. The Profile Line. A plane which is perpendicular to the ground line is known as a **profile plane**, and any line lying in a profile plane is termed a **profile line.** We have just seen that such a line cannot be projected in the same way as the general straight line. Hence problems in which such lines occur will usually — but not always — call for particular solutions. The simplest method of dealing with a profile line is usually by means of an additional plane or projection. The solution of cases involving such lines will be deferred, in general, until Chapter VI, where this topic is considered.

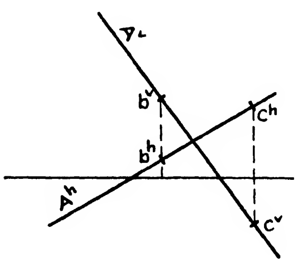

FIG. 51.

36. A Point Lying in a Line (Fig. 51). It follows at once from the second proposition of § 28, that if a point lies in a line, the projections of the point lie in the projections of the line. This condition is sufficient if the line does not lie in a profile plane. (See § 35.)

37. Traces of a Line. Of all the points in a straight line, the two in which it pierces the planes of projection are considered the most important. These points are called the **traces** of the line. The horizontal trace is the point in which the line pierces H, and will be designated as the point s; the vertical trace, in which the line pierces V, will be called t.

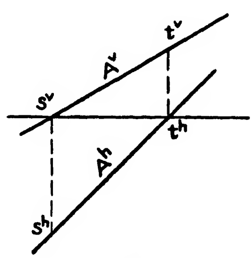

FIG. 52.

Problem 1. ***To find the traces of a straight line.***

Analysis. The solution of this problem depends on direct visualization.

General Case. Line not lying in a profile plane.

Construction (Figs. 52, 53). *The horizontal trace.* Looking towards V, H is seen edgewise as GL, the line A appears as A^v, hence the line A will pierce H at the point seen as s^v, where A^v crosses GL. While this projection conveys no idea of the distance of the point in H from V, it does single out, to

the exclusion of every other point, the H piercing point of the line. The actual position of the point in H is found at s^h by the projector $s^v s^h$.

The vertical trace. Looking down toward H, V is seen edgewise as GL, A is seen as A^h, hence A is seen to pass through V at the point t^h. The actual location, t^v, in V, is found on A^v by means of the projector $t^h t^v$.

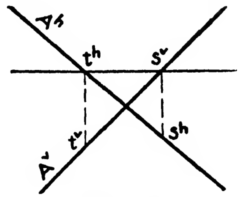

FIG. 53.

Special Case. A profile line (no figure). The general solution evidently fails; the solution will be given in Chapter VI.

Problem 2. ***Given the two traces of a straight line, to find its projections.***

Analysis. Each trace is a point lying in a coördinate plane; and for such a point, one projection is identical with the point, the other projection is in the ground line.

Construction (Figs. 52 and 53). The traces s^h and t^v are given; s^v and t^h are found by projection on GL; s^h and t^h determine A^h, s^v and t^v determine A^v.

If s^h and t^v lie in the same projector (Fig. 54), the required line is a profile line. Note that this is a definite line, since two points, s and t, are known.

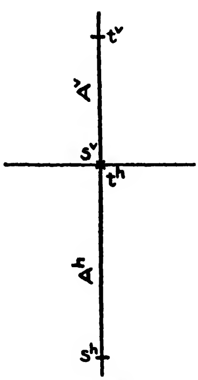

FIG. 54.

38. Exceptional Cases. The solution to Problem 2 shows that, in general, a straight line is uniquely determined by its two traces. The only exception is the case in which s and t fall together as a single point on the ground line. The required line can then be any line which passes through this point; in order to determine the line, some other condition must be given.

The following questions on traces are left to the student:

1. How must a line be placed so as to have but one trace?
2. How must a line be placed so as to have no trace?
3. How must a line be placed so that its horizontal (or vertical) trace is indeterminate?

CHAPTER IV

SIMPLE SHADOWS

39. Shadows. The **shadow** of a point on a plane is the point in which a ray of light passing through the given point is intercepted by the plane.

The conventional ray of light is a line sloping downward, backward, to the right, with both projections inclined at 45° with the ground line. Such a ray is shown by the line L, Fig. 55.

40. Shadow of a Point. The shadow on V of the point a, Fig. 55, is the point, a_t, in which the ray of light, L, passing through a, intersects V. The shadow on H is the point, a_s, where L pierces H.

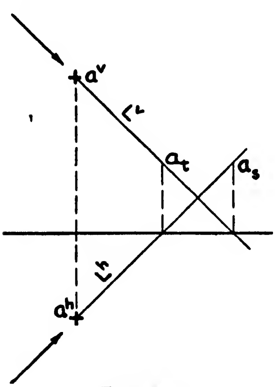

FIG. 55.

It will be noted that the points a_t and a_s are the traces (§ 37) of the line L on V and H respectively. Hence the shadow of a point on the coördinate planes is found by drawing through the point a line representing the ray of light, and then finding the traces of this line.

41. Opaque and Transparent Planes. In order that an actual shadow may be cast, it is necessary that the plane receiving the shadow be opaque. Thus, in Fig. 55, if V be considered opaque, a_t is the actual shadow of the point a; but if V be considered transparent and H opaque, a_s will be the actual shadow. In work with shadows, it is customary to place the object in the first quadrant, and to consider both H and V as opaque.

Since the two projections of the ray of light make equal angles with the ground line, it follows that the actual shadow

of a point must always fall on the nearer coördinate plane. In the case of a solid object, the actual shadow may fall wholly on *H*, wholly on *V*, or partly on both planes, according to the size and position of the object.

42. Shadow of a Line. The shadow of a line will consist of the shadows of the points which compose the line. If the line is straight, only the shadows of the two ends of the line need be cast. In Fig. 56, the actual shadows of both points a and b fall on V; hence the shadow of the line ab is the line a_tb_t, lying

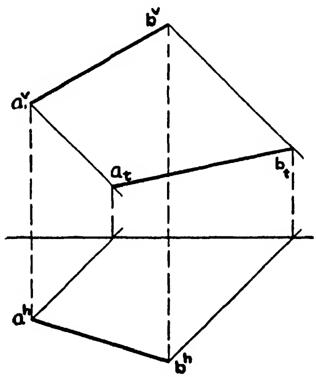

FIG. 56.

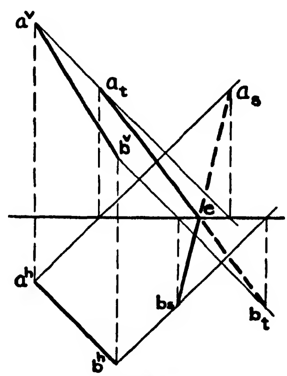

FIG. 57.

wholly in V. In Fig. 57, the actual shadow falls partly on H and partly on V. To find this shadow, we may find the complete shadow on each plane, regarding the other plane as transparent, and then take the actual portion of each shadow. But the two shadows, a_sb_s and a_tb_t, one in H and one in V, must intersect in a point, e, in GL, since this line is the intersection of the planes H and V. Hence we may find one complete shadow, as a_sb_s, note the point e in which a_sb_s crosses GL, and draw from e to the actual shadow a_t.

43. Shadows of Surfaces. Shadows of surfaces and solids are found by extending the methods just given for the point and straight line. The only other case which will be taken up at this time is that of a convex solid bounded by plane faces.

44. Shadow of a Convex Solid. A frustum of an irregular triangular pyramid (Fig. 58) has six corners and nine rectilinear edges. Not all of them cast shadows which lie on the boundary of the complete shadow. Find the shadows of the six corners 1, 2 ... 6 on one of the planes, as *H*, and draw the largest convex

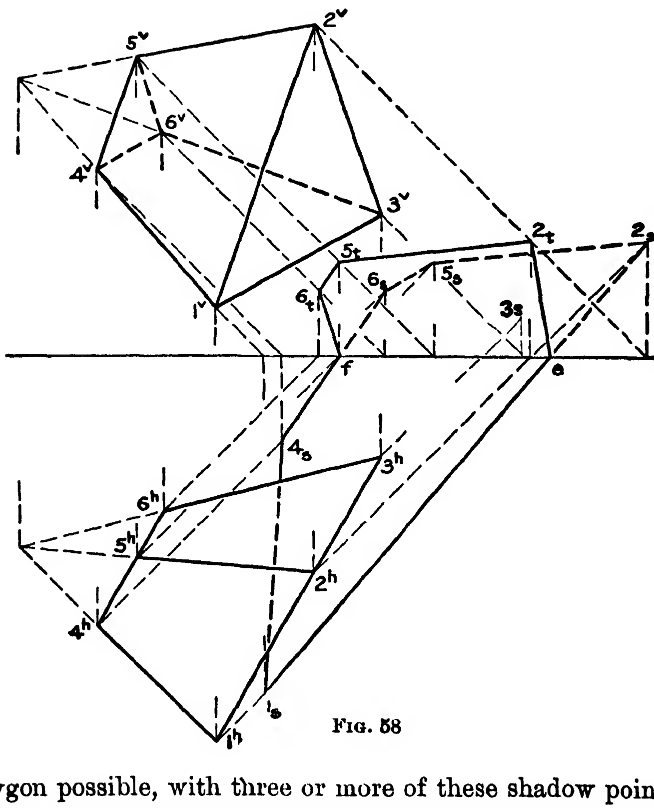

FIG. 58

polygon possible, with three or more of these shadow points as vertices. In Fig. 58, the shadow is the polygon $1_s 2_s 5_s 6_s 4_s 1_s$, the point 3_s being omitted because it falls inside the polygon. The part $1_s e f 4_s$ on *H* in front of *GL* is an actual shadow; the remaining part, $e 2_s 5_s 6_s f$, shows the portion of the shadow intercepted by *V*. Find the actual shadows on *V* of the points 2, 5, and 6; then $e 2_t 5_t 6_t f$ is the part which falls on *V*, and completes the actual shadow.

CHAPTER V

REPRESENTATION OF THE PLANE

45. Representation of the Plane. It has been shown (§ 28) that a straight line of indefinite extent can be represented by two orthographic projections on *H* and on *V*.

This method, however, cannot be extended to the representation of a plane of indefinite extent, since the projections of all the points in the plane would, in general, cover the whole plane of projection.

It has been shown, however (§ 38), that a line of indefinite extent may also be determined by its points of intersection with the planes of projection.

This method is capable of extension to the representation of a plane.

46. Traces of a Plane. A plane of indefinite extent will, in general, cut *H* and *V* in two straight lines, which meet in the point in which the given plane cuts the ground line.

These lines of intersection of the plane with *H* and *V* are called, respectively, the **horizontal** and **vertical traces** of the given plane.

Conversely, two arbitrary straight lines, one in *H* and one in *V*, which intersect the ground line at the same point, determine a plane of which the arbitrary lines are the horizontal and vertical traces.

Thus a definite method of representation of the plane has been found.

This method of representation fails only in the case in which the plane contains the ground line, when the two traces will coincide with each other and with the ground line. In this case the plane will not be determined until some additional condition is imposed.

Compare this with the analogous case of a line having coincident traces, § 38.

47. Notation of the Plane. A plane in space will be denoted by a single capital letter, taken usually from the middle or end of the alphabet.

The horizontal trace of a plane Q, being the line of intersection of H and Q, will be called HQ.

The vertical trace, or intersection of V and Q, will be called VQ.

The form HQ is used in preference to the form QH, since the latter, in speech, might be confused with Q^h, the H-projection of a line Q, which is by no means the same thing as the H-trace of a plane Q.

48. Visualization of the Plane. The visualization of a plane from its traces is an entirely different process from reading the projections of a line or of a point. The traces of a plane oblique to H and V are not projections of the plane in the ordinary sense.

Let HQ and VQ, Fig. 59, be the horizontal and vertical traces of a given plane Q.

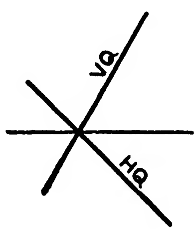

Fig. 59.

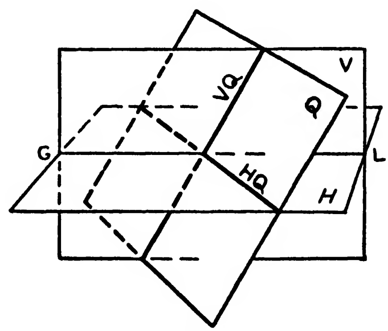

Fig. 60.

Since the plane is indefinite in extent and cuts the ground line, it must pass into all four quadrants.

Since the two traces are oblique to GL, it follows that the plane determined by them is oblique to both H and V.

49. The Slope of a Plane. The slope of an oblique plane is described in the same terms as the slope of a line, and may perhaps be best understood by reference to the planes of Figs. 59–63.

Plane Q in Fig. 59, shown pictorially in Fig. 60, would be said to slope downward, forward, and to the left (or upward, backward, to the right).

In Fig. 61 the slope is downward, backward, to the left (abbreviated to D.B.L.).

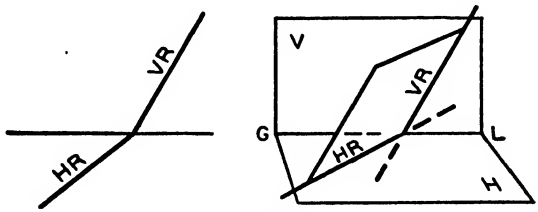

FIG. 61.

In Fig. 62, the slope is downward, forward, to the right (D.F.R.).

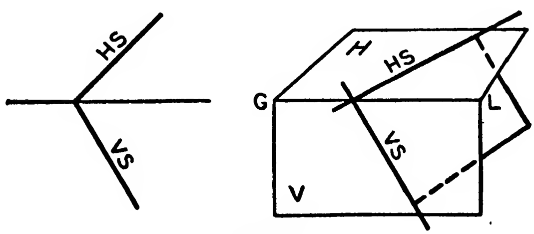

FIG. 62.

In Fig. 63, plane Q, perpendicular to H, would be said to slope simply backward to the right (B.R.).

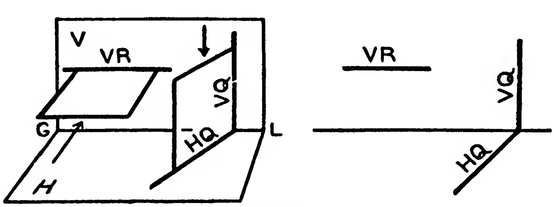

FIG. 63.

It is usually easier to visualize the slope of a plane by considering, at first, one quadrant only (the first or third). The plane can then be visualized as extending into the other quadrants.

50. Special Positions of the Plane. The student should satisfy himself of the truth of the following propositions by direct visualization.

1. *If a plane, Q, is parallel to H, there is no H-trace* (i.e. *HQ does not exist*), *while VQ is parallel to the ground line.*

2. *Conversely, if Q has no H-trace, VQ must be parallel to the ground line, and Q is parallel to H.*

3. *Similarly, if Q is parallel to V, VQ does not exist, while HQ is parallel to the ground line; and conversely.*

4. *If Q is parallel to the ground line, HQ and VQ are also parallel to the ground line; and conversely.*

5. *If Q is perpendicular to H, VQ is perpendicular to the ground line; if Q is perpendicular to V, HQ is perpendicular to the ground line; and conversely in each case.*

6. *If Q is perpendicular to both H and V, that is, perpendicular to the ground line, HQ and VQ fall together in the same perpendicular to GL.*

51. Edge View of a Plane. The particular position of a plane in which it becomes perpendicular to one of the coördinate

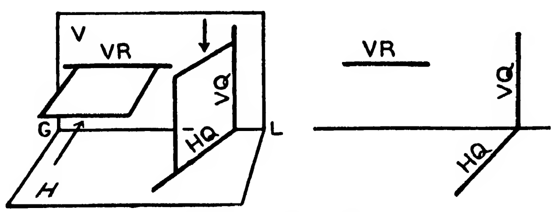

FIG. 63 (repeated).

planes is an important one to visualize. If Q is perpendicular to H, then the H-projection of every point in Q must fall in HQ, which thus becomes an actual *projection* of the plane. The trace HQ is, in fact, an **edge view** of the plane Q (Fig. 63). It should be noted that a plane parallel to one of the coördinate planes is necessarily perpendicular to the other, and that the single trace of such a plane is its edge view (plane R, Fig. 63).

CHAPTER VI

THE PROFILE PLANE OF PROJECTION

52. The Profile Plane. The profile plane, when used as a plane of projection, will be designated by the letter *P*. This plane has already been defined as a plane perpendicular to the ground line (§ 35), that is, *P* is perpendicular to both *H* and *V*. Hence the three planes *H*, *V*, and *P* form a system of mutually perpendicular planes (Fig. 64), intersecting in a

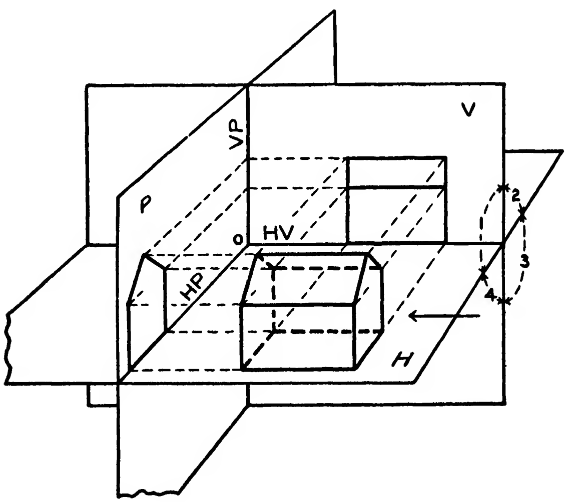

FIG. 64.

common point *o*, and intersecting each other by pairs in three mutually perpendicular lines, *HV*, *HP*, and *VP*, all passing through *o*. The intersection, *HV*, of *H* and *V* is the ground line.

53. A Profile Projection. Let the model of some familiar object, such as a cabin, be placed on the horizontal plane in the first quadrant, Fig. 64. Let the front and back be parallel to the *V*-plane, and the ends parallel to *P*. Let a view be taken

looking in the direction of the arrow, that is, from right to left. In such a view, the whole surface of *P* will be seen, while the

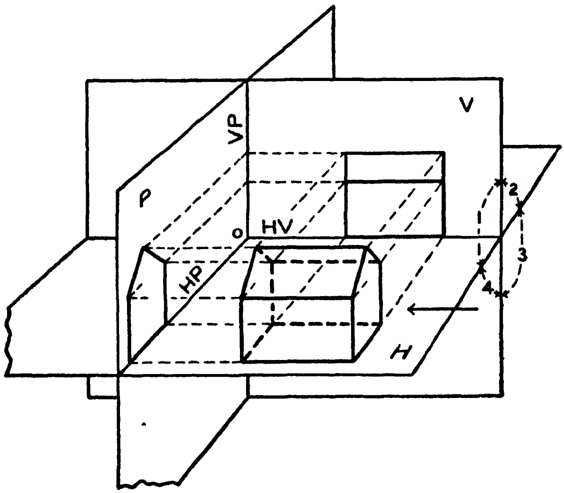

FIG. 64 (repeated).

V-plane will be seen edgewise as a vertical line, the *H*-plane edgewise as a horizontal line, and only the right-hand end of the cabin will appear.

The actual end view is shown in Fig. 65. No boundary is shown in this figure for the *P*-plane, since it, like the *H*- and

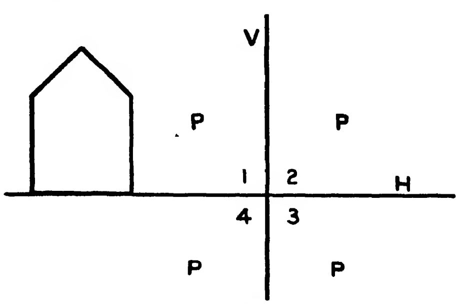

FIG. 65.

V-planes, is of indefinite extent. The quadrants will appear as indicated by the numbers, and must always so appear when *P* is viewed from the right. In this end or profile view of

the cabin, both heights and widths, or, in other words, all distances perpendicular to *H* or to *V* are shown in their true size. Hence in this view the actual distances from the object to *H* and to *V* are necessarily shown.

54. Relation of the Profile Projection to the *H*- and *V*-Projections. Figure 66 shows the construction of the end view, or

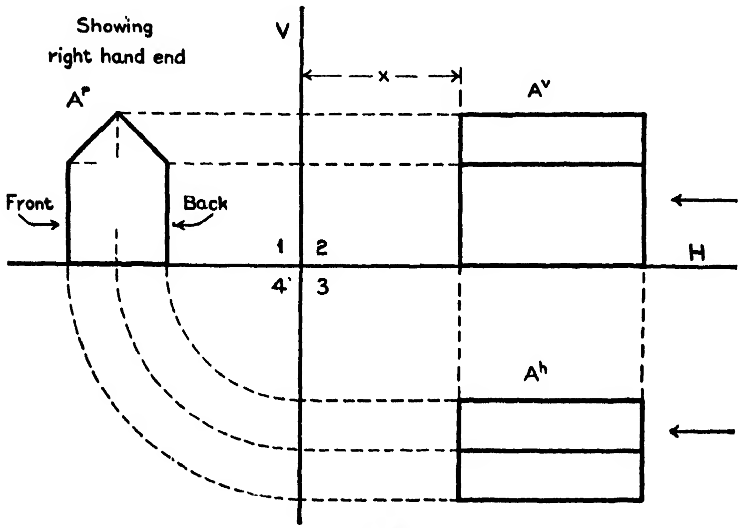

FIG. 66.

profile projection, of the cabin from given horizontal and vertical projections. The distance x, from the edge view of *V* to the end of the cabin, may be chosen at will, as this evidently has no effect on the shape or size of the end view of the cabin, or its distances from *H* and *V*.

The edge views of *H* and *V*, together with the end view of the cabin, might also be placed at the right of the plan and elevation, instead of the left; and would preferably be so placed if the object were given in the third quadrant. This would locate the views in accordance with the usual custom of placing the view of the right-hand end of an object to the right of its front view. For a consideration of a view of the left-hand end, see §§ 62 and 63.

55. A Second Example. Let it be required to draw the plan, elevation, and side elevation (profile projection) of a piano packing case placed in the third quadrant, with its long edges parallel to both *H* and *V*. The dimensions of the case and its distances from *H* and *V* are supposed to be

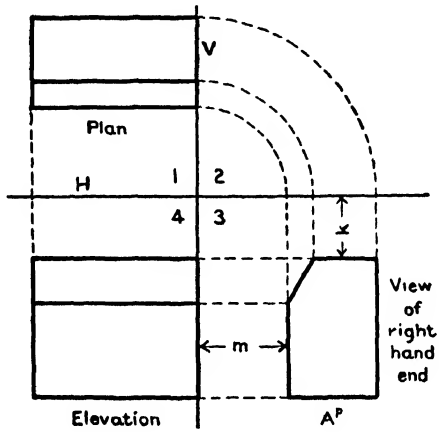

FIG. 67.

known. The end view is drawn first at A^p (Fig. 67), located as shown in the third quadrant, and at the given distances, *k* from *H* and *m* from *V*. The plan and elevation may then be readily constructed as shown. It is important to see clearly why all the long edges appear visible both in plan and elevation.

56. Profile Projections in All Four Quadrants. Figures 68–71 show a regular triangular prism placed successively in the four quadrants. The long edges of the prism are parallel to both *H* and *V*, and in each quadrant the upper front lateral face makes an angle of 15° with *H*. In each position the right-hand end is shown in the profile view. The student should visualize the prism for each of the four positions, noting especially the visibility of the edges lettered B^v and C^h, lettered also in the profile view as the points B^p and C^p.

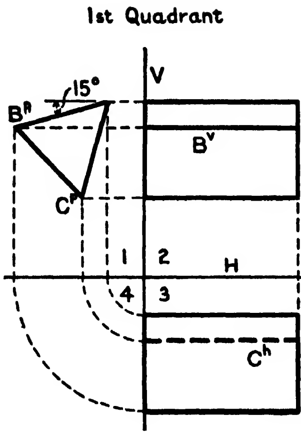

Fig. 68.

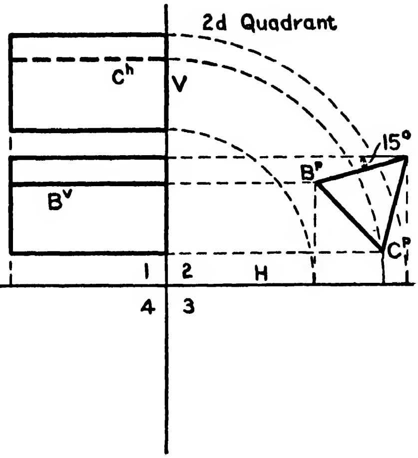

Fig. 69.

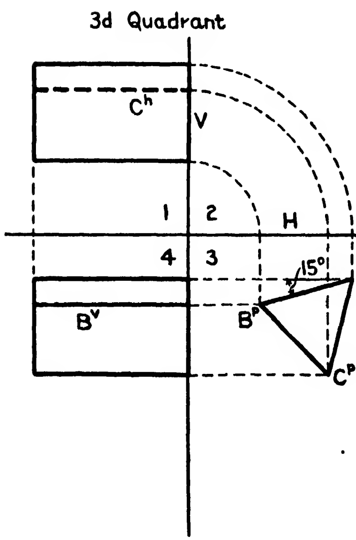

Fig. 70.

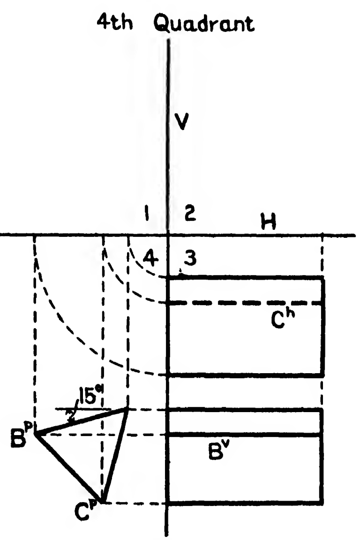

Fig. 71.

57. Profile Projection of the Straight Line. The profile projection of any straight line may be obtained from its horizontal and vertical projections by finding the profile projections of any two points in the line (Fig. 72).

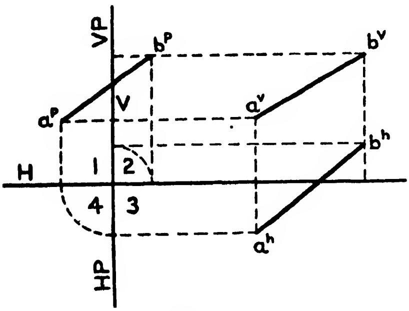

FIG. 72.

An important case of the profile projection of a line arises when the line is parallel to P, that is, when the line is a profile line. By means of its profile projection we can solve the following problems for such a line, which cover solutions previously omitted (§§ 36, 37).

58. Problems on the Profile Line. *Given the horizontal and vertical projections of a profile line, to find*

(*a*) *its true length ;*

(*b*) *its traces on H and V ;*

(*c*) *the projections of a point lying in the line.*

(*a*) *THE TRUE LENGTH.*

Analysis. Since the line is parallel to P, its profile projection will be equal to its true length.

(*b*) *THE TRACES ON H AND V.*

Analysis. These traces will appear in the profile projection as the points in which the line cuts the respective edge views of H and V.

Construction for (*a*) **and** (*b*) (Fig. 73). Let $ab(a^hb^h, a^vb^v)$ be the given profile line. If the profile plane of projection is not given, it may be assumed anywhere, perpendicular to the ground line. Find the profile projections of the points a and b. Then a^pb^p is the profile projection of the line, and is equal to the true length of ab. Produce a^pb^p indefinitely. Then the line is seen to pierce H at s^p, and V at t^p. Find the horizontal and vertical projections of the points s and t; the point $s(s^h, s^v)$ is the H-trace, and the point $t(t^h, t^v)$ is the V-trace.

A second example is shown in Fig. 74. This is the more common construction, the profile plane of projection being taken directly through the given line.

(c) *THE PROJECTIONS OF A POINT ON THE LINE.*

Analysis. If the profile projection of the line be found, we

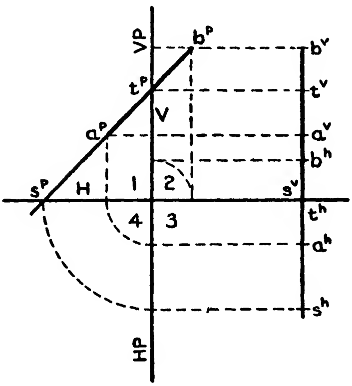

FIG. 73.

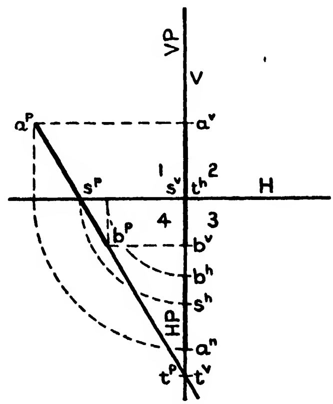

FIG. 74.

can project a point in the H-projection to the P-projection, and from this to the V-projection; or vice versa.

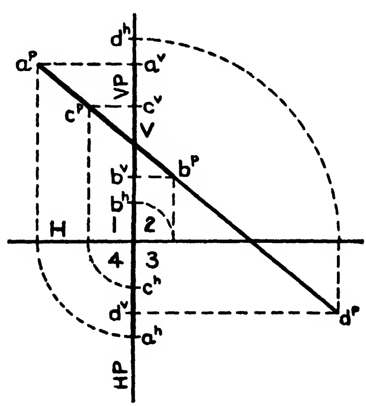

FIG. 75.

Construction (Fig. 75). Let *ab* (a^hb^h, a^vb^v) be the given line. Find a^pb^p. Now if c^h is given, project from c^h to c^p in a^pb^p; and from c^p to c^v in a^vb^v. If c^v is given, c^h may be found by reversing the process.

The line *ab* may be produced indefinitely, and the assumed point may be beyond the limits of *a* and *b*. Thus, if d^v is taken, d^h is found as shown, and the point *d* lies on the line produced into the third quadrant. Or, d^h may be given, and d^v found.

59. Profile Trace of a Line. The general line, oblique to H, V, and P, will pierce all these planes. Hence, besides the horizontal trace s, and the vertical trace t, it will have a profile trace, u, which must lie in the profile projection of the line.

Given the horizontal and vertical projections of a general line, to find its profile projection and profile trace.

Analysis. The profile projection of any straight line can be obtained from its horizontal and vertical projections by finding the profile projections of any two points in the line (§ 57). The profile trace can be found by noting the point in which the given line pierces the edge view of P.

Construction (Fig. 76). A^h and A^v are the given projections. Find the points s and t, the H- and V-traces of the line A (Prob. 1, § 37). Project s to s^p in GL, and t to t^p in VP; s^p and t^p determine the required profile projection A^p. Since

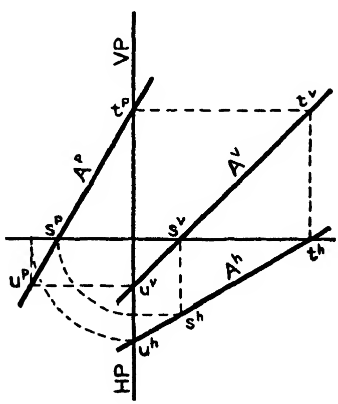

Fig. 76.

HP and VP are both edge views of P, we have u^h and u^v as the H- and V-projections of the point u; from u^h and u^v find u^p, the actual profile trace. As a check, u^p must lie in A^p.

Since any two of its projections are sufficient to determine the line A, we have the following proposition:

Corollary. *Given A^p and either A^h or A^v; to find the remaining projection and the three traces of the line.*

Analysis. The desired result may be obtained by reversing the preceding construction.

The construction is left to the student.

60. Profile Trace of a Plane. A plane not parallel to *P* will intersect it, and the line of intersection is known as the profile trace of the plane.

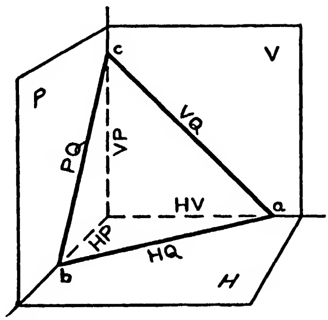

FIG. 77.

A general plane *Q* is shown in Fig. 77, intersecting *H*, *V*, and *P*, in the traces *HQ*, *VQ*, and *PQ*. It is evident that *HQ* and *VQ* intersect on *HV* (the intersection of *H* and *V*, or ground line), *HQ* and *PQ* intersect on *HP*, and *VQ* and *PQ* intersect on *VP*. Whence, if *HQ* and *VQ* are given, *PQ* may be found as shown in Fig. 78.

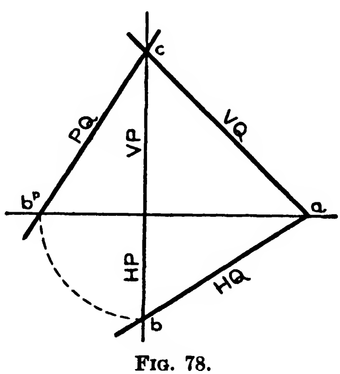

FIG. 78.

The profile trace of a different plane, *R*, is found in Fig 79.

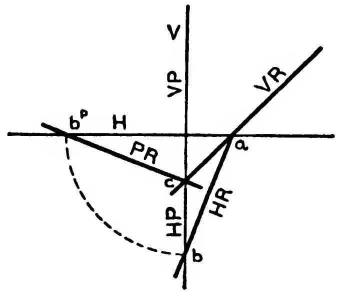

FIG. 79.

The profile trace of a plane parallel to the ground line is found in the same way, as shown in Fig. 80. This is an im-

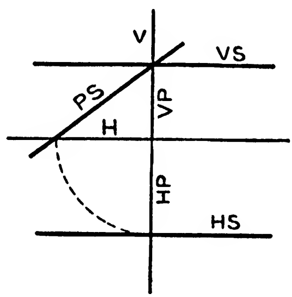

FIG. 80.

portant case, for the plane *S* is perpendicular to *P*, and consequently the profile trace is an *edge view* of the plane *S*.

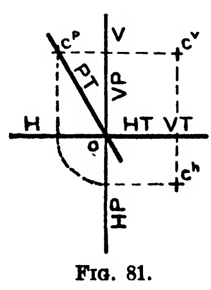

FIG. 81.

In Fig. 81 the given plane *T* passes through the ground line,

and is determined by the coincident traces *HT*, *VT*, and the point *c* lying in the plane. To find the profile trace *PT*, find first the profile projection c^p, of the point *c*. Then, as in the preceding case, *PT* is an edge view of *T*, and therefore passes through c^p and *o*. It will be seen that the plane *T* is definitely determined by its three traces *HT*, *VT*, and *PT*.

Instead of being determined by passing through the point *c*, the plane *T* might also have been determined by stating the quadrants through which it passes, and the angle between *T* and either *H* or *V*. Then the profile trace, *PT*, could have been located accordingly.

From Figs. 78–81 it is evident that the three traces of a plane are so related that if any two are given, the third can be found. The cases in which the given traces are the profile trace and either the horizontal or vertical trace, to find the remaining trace, are left to the student.

61. Planes Parallel to *H* or *V*. A plane parallel to *H* or *V* has no trace on one of these planes, and the student sometimes has difficulty in constructing the profile trace. Such a plane may be regarded as a limiting case of the plane *S*, Fig. 80; but the result may be obtained by direct visualization, as follows. Since the plane is perpendicular to *P*, the profile trace will be its edge view. Then the profile projection shows the

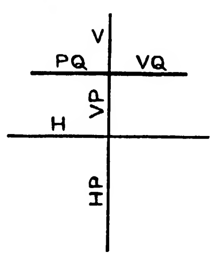

Fig. 82.

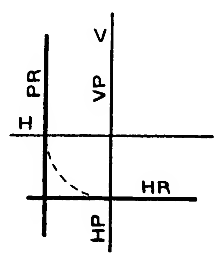

Fig. 83.

edge views of both *H* and *V*, to the proper one of which the plane will show parallel.

A plane parallel to *H* is shown in Fig. 82; one parallel to *V* in Fig. 83.

62. Left-side Views. Hitherto, the profile projection has been obtained in just one way. Hence the student might suppose that the method of projecting on the profile plane is as invariable as that of projecting on *H* or *V*. In practice, however, this is not the case. The method thus far adopted, when applied to an actual object, always gives a view of its right side; whereas a view of the left side may be decidedly prefer-

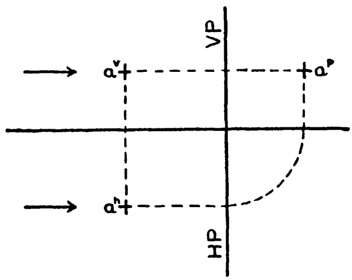

FIG. 84.

able. For this purpose, *P* is viewed from its left side, that is, from left to right, and then projected so as to bring this side into the drawing surface. This method is shown for a point in the first quadrant in Fig. 84. That this method is directly opposite to that previously used is shown also by the line *ab*, Fig. 85. Since the *H*- and *V*-projections of *a* and *b* show that

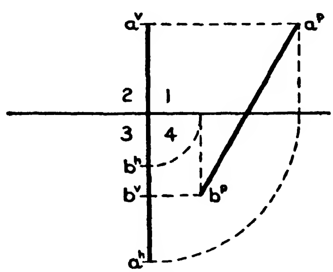

FIG. 85.

these points are in the first and fourth quadrants respectively, the quadrants must appear in the profile projection as marked.

In Fig. 86 the profile trace of the same plane *Q* is obtained, first, by looking from right to left, and then by looking from

left to right. Note that, since *HQ* is at 45° with the ground line, *PQ* coincides with *VQ* in the first figure, and that the

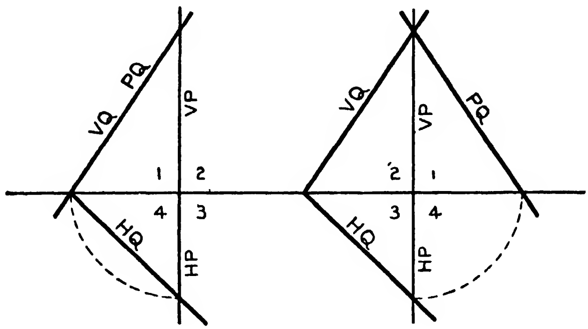

FIG. 86.

second figure is much the clearer, especially if some further construction is to be made with the plane *Q*.

Hereafter we shall view *P* from either side; although, other things being equal, the direction from right to left will be preferred. If the quadrants are numbered in every profile projection, there should be no confusion, whichever view is used.

63. Conventional Placing of Left-side Views. When the profile projection of a solid object is made so as to show its left side, it is customary to place this projection at the right of the front elevation when the object is located in the first quadrant, and at the left of the front elevation when the object is located in the third quadrant. This is the reverse of the method of placing the view of the right side, as shown in Figs. 68 and 70. (See § 54.) Actual objects are rarely placed in the second or fourth quadrants, so that there is no fixed custom for locating such views.

64. Second Method of Obtaining Profile Projections. Profile projections may be constructed also by treating the profile plane as a secondary plane of projection, by the methods of the next chapter. Such a construction is less common, however, than that given above, since the resulting position of the profile projection is not so generally satisfactory.

CHAPTER VII

SECONDARY PLANES OF PROJECTION

65. Secondary Planes of Projection. In practical work, views of objects are often wanted in other directions than those which are obtained by the use of the horizontal and vertical coördinate planes. In fact, it often happens that an actual object cannot be adequately represented by a simple plan and elevation. In the theory, also, an additional projection in a suitably chosen direction may give readily a solution otherwise difficult to obtain. Such views, or projections, are obtained on secondary planes of projection, taken perpendicular to either *H* or *V*, and making any angle whatever with the other coördinate plane.

Secondary planes of projection are by preference taken perpendicular to *H*, for while any plane perpendicular to *H* is vertical, a plane perpendicular to *V* and oblique to *H* is neither vertical nor horizontal, and therefore does not conform to either of the natural directions (§ 6).

66. The Profile Plane as a Secondary Plane. A very important secondary plane of projection is the profile plane, which is perpendicular to both *H* and *V*. This plane has just been discussed in detail in the preceding chapter. Indeed, some writers consider this plane as a third coördinate plane of equal rank with the horizontal and vertical coördinate planes. In the present chapter, the profile plane will be considered as a special case of a plane perpendicular to *H*.

67. Secondary Ground Lines. The method of projecting on a secondary vertical plane is shown in Figs. 87 and 88. The secondary plane V_1 intersects *H* in a secondary ground line, G_1L_1, which may be at any angle with the original ground line *GL*. Since the projectors aa^v and aa_1^v, Fig. 87, are both parallel to *H*, the distances a^ve and $a_1^ve_1$ are equal, both being equal to aa^h, the distance of the point from *H*.

68. Principles of Secondary Projections. In general for any secondary vertical plane of projection we have the following propositions:

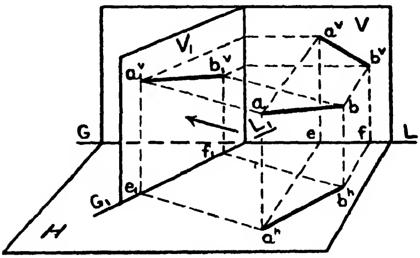

Fig. 87.

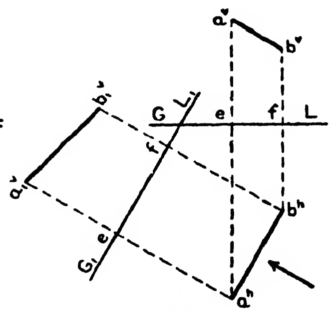

Fig. 88.

1. *The plane may be viewed from either side, irrespective of the position of the object, but the direction of sight must always be at right angles to the plane* (§ 5), *and the projectors must be drawn at right angles to* G_1L_1.

2. *Distances above H must be laid off on the farther side of* G_1L_1, *and distances below H on the near side.*

3. *All vertical distances, either above or below H, remain unchanged.*

4. *Hence, if we have given the plan and elevation of any point, whether it be an isolated point, one end of a line, or a corner of a solid, the secondary V-projection may be obtained as follows:*

(*a*) Decide from which side the secondary *V*-plane is to be viewed.

(*b*) Draw a projector from the *H*-projection of the point perpendicular to the secondary ground line.

(*c*) Take the distance of the point from *H*, and mark it off on the farther side of G_1L_1 if the point is above *H*, and on the near side if the point is below. Thus, in Fig. 87, V_1 is viewed in the direction of the arrow, and the projection $a_1^v b_1^v$ can be laid off only on the side of G_1L_1 as shown. Conversely, if the secondary projection $a_1^v b_1^v$ is constructed as in Fig. 88, it follows that V_1 is viewed from the side indicated by the arrow, Fig. 87.

69. Additional Examples. Additional examples of projections on a secondary V-plane are given in Figs. 89 and 90. In

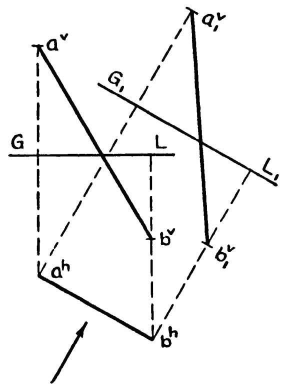

Fig. 89.

Fig. 89, note that the directions of the points a and b from H, as shown by the relations of a^v and b^v to GL, must be preserved in the secondary projection. In Fig. 90 a profile line is shown projected on a profile plane by this method. (See §§ 64, 66.)

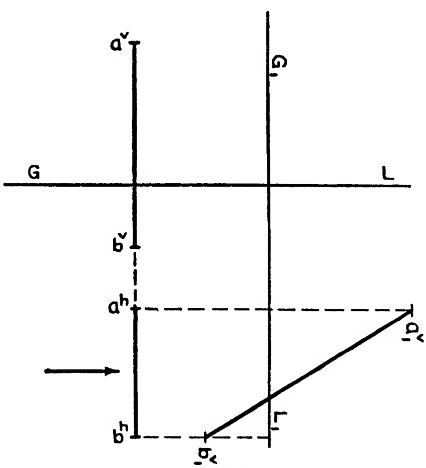

Fig. 90.

70. Simplification of Problems by Means of Secondary Projections. In the solution of a problem, there is no advantage in introducing a secondary plane of projection unless the new projection is in some way simpler than the original projections. A point always projects as a point, and cannot be made any simpler.

The simplest projection of a straight line is a point. This projection can be obtained by a single secondary plane of projection only when the given line is parallel to one of the original coördinate planes. Let the line A (Fig. 91) be parallel to H;

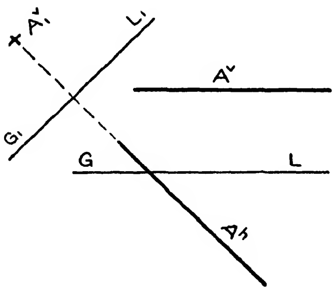

Fig. 91.

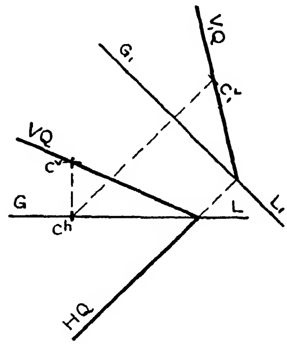

Fig. 92.

then if G_1L_1 is taken perpendicular to A^h, the line A will be seen endwise, and the projection $A_1{}^v$ will become a point.

Another simple and useful projection of a straight line results when the line is parallel to a plane of projection. This can always be attained by means of a secondary V-plane, by taking G_1L_1 parallel to the H-projection of the line, as shown in Figs. 88, 89, and 90.

The simplest position of a plane is that in which one trace is an edge view of the plane. Such a view can always be obtained by means of a secondary vertical plane of projection. Let Q, Fig. 92, be a general plane. Assume a secondary ground line G_1L_1 perpendicular to HQ. Then, since HQ is perpendicular to this ground line, Q is perpendicular to V_1 (§ 50, par. 5), and its trace, V_1Q, on the secondary V-plane is an edge view of Q (§ 51). To find V_1Q: since Q is seen edgewise against V_1, V_1Q will pass through the projection of any point in Q.

Now VQ, being the intersection of V and Q, lies in both V and Q. Assume any point c^v in VQ; then the point c is in Q and also in V, whence c^h is in GL. Using the projections c^h and c^v, find the secondary projection $c_1{}^v$. The required trace,

V_1Q, passes through $c_1{}^v$ and the point in which HQ intersects the ground line G_1L_1.

71. Projections of Solids. A secondary projection of a solid is obtained by extending the method used for points and lines. As an illustration, the irregular triangular pyramid of Fig. 32 is shown in Fig. 93 projected on a secondary vertical plane. In this illustration no attempt was made to simplify the projection of the solid, nor is it always possible to do so. A case in which a simplification is effected, however, is shown in Fig. 94;

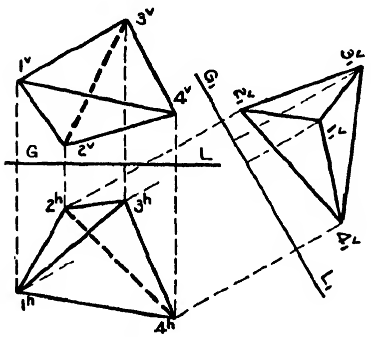

FIG. 93.

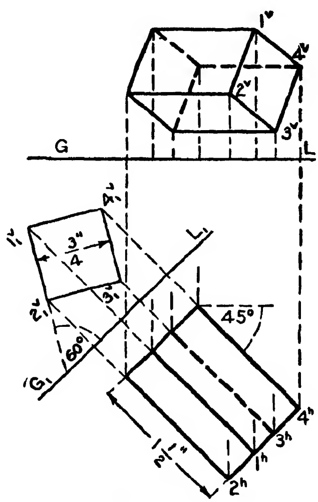

FIG. 94.

here the projection on the secondary V-plane gives information about the size, shape, and position of the prism which is not directly obtainable from the original V-projection.

72. The Two V-Projections Compared. The relation between two V-projections of an object is the same, whichever be considered the original, and which the secondary, projection (§ 67). Hence, in Fig. 94, instead of constructing the V_1-projection from the H- and V-projections, the V-projection might be constructed if the H- and V_1-projections are known. That is, if an object is known to have a simple projection when seen from

a certain point of view, this fact may be used to construct the projections of the object. The student should examine Fig. 94, assuming it to be drawn as the answer to the following problem:

Draw the *H*- and *V*-projections of a square right prism. The edges of the base are $\frac{3}{4}''$; length of prism $1\frac{1}{2}''$. The lateral edges are parallel to *H*, 45° with *V*, backward to the left; the front end of the highest lateral edge is located at point 1 (1^h, 1^v). The front lateral face of the prism makes 60° with *H*.

[HINT. After point 1 is located in all three projections, the views are completed in this order: V_1-projection, *H*-projection, *V*-projection.]

The object shown in Fig. 95 consists of two circular conical

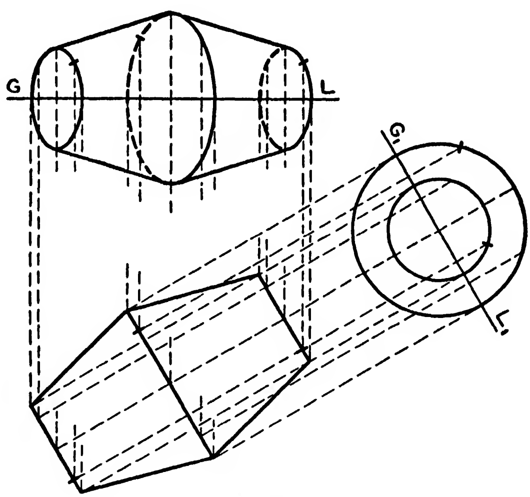

FIG. 95.

frustums placed base to base. Note the relation between the two *V*-projections, and especially the position of the ground line.

73. Oblique Secondary Planes. Secondary planes of projection perpendicular to *V* and oblique to *H* will not be considered in detail, as their use in practical work is very limited. (See § 65.) The treatment being entirely analogous to the plane perpendicular to *H*, the student should have no difficulty in using such a plane of projection should occasion arise.

CHAPTER VIII

REVOLUTION OF A POINT — TRUE LENGTH OF A STRAIGHT LINE — APPLICATIONS

74. Revolution of a Point about a Straight Line. A point may be revolved about any straight line as an axis through a complete revolution or through a portion of a revolution. In the case of a complete revolution, the path of the revolving point will be the circumference of a circle which lies in a plane passing through the point and perpendicular to the axis and whose

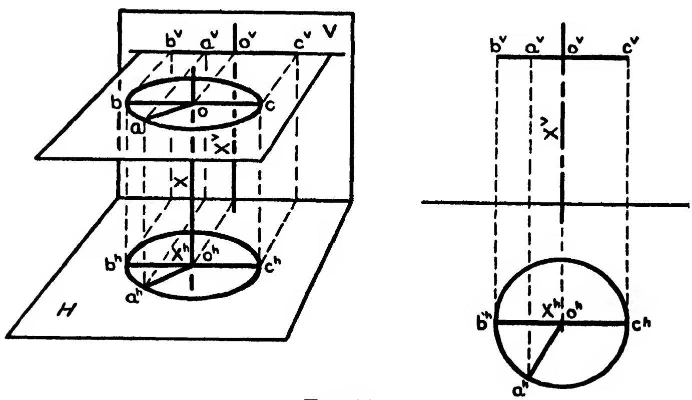

FIG. 96.

center lies on the axis. In the case of a portion of a revolution, the path will be an arc of such a circle.

We shall proceed to discuss certain simple cases in which the axis of revolution lies in a particular position.

75. Axes of Revolution Perpendicular to *H* or *V*. In certain positions of the axis of revolution, the circular path of the revolving point will have simple projections on either *H* or *V*, or both.

Let a, Fig. 96, be revolved about an axis perpendicular to

H. The circle in which *a* revolves will be parallel to *H*, and will project on *H* in its true shape and size, while the *V*-projection will be a straight line parallel to the ground line. Similarly (Fig. 97), if the point is revolved about an axis perpendicular to *V*, the circular path projects on *V* as a circle, and on *H* as a straight line parallel to *GL*.

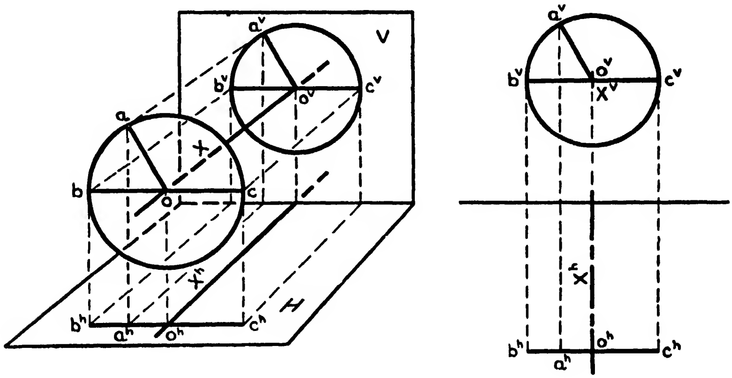

FIG. 97.

76. Revolution of a Point about an Axis Lying in a Given Plane (Fig. 98). Let us consider a point *a*, which is to be revolved about an axis *X*, lying in a plane *Q*. The circular path of the revolving point will project on *Q* as a straight line, *bc*, perpendicular to the axis *X*. It follows that if the point *a* be revolved into the plane *Q*, its revolved position, *b* or *c*, will be at a distance *be* or *ce* from the axis *X*, equal to the true distance *ae*, of the point from the axis.

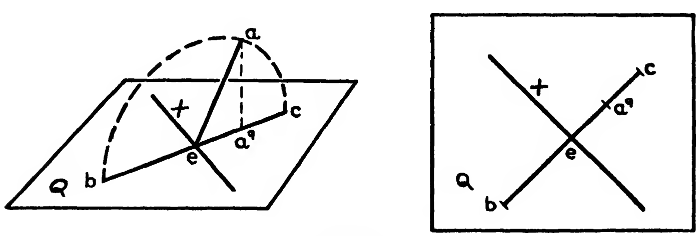

FIG. 98.

77. The True Length of a Line. Let *ab* be a straight line of definite length. If placed parallel to any plane of projection, as Q, Fig. 99, the projection of $a^q b^q$ will evidently be equal in length to *ab*. If, however, the line is placed at any angle with Q, the length of the projection will be shorter than that of the line. (See the line *cd*, Fig. 39.) But by revolution about a

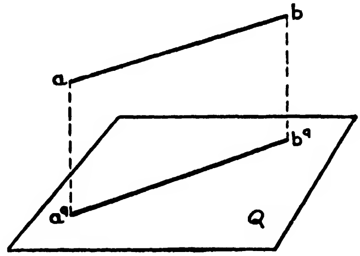

FIG. 99.

suitable axis, a line can always be brought into such a position that the true length will appear.

78. First Method for Finding True Length. The projection of a line is equal to its true length when the line is parallel to the plane of projection (§ 77). The true length of any straight line may therefore be found by revolving the line until it is parallel to either H or V.

Problem 3. ***To find the true length of a straight line.***

FIRST METHOD. By revolving about an axis perpendicular to H or V.

Analysis. Through either end of the line assume an axis perpendicular to H and (of necessity) parallel to V. Revolve the given line about this axis until the line is parallel to V. Then the V-projection of the line in its revolved position will show the true length of the line.

Or, assume the axis perpendicular to V and hence parallel to H, and revolve the given line until parallel to H.

Construction (Fig. 100). Let *ab* ($a^h b^h$, $a^v b^v$) be the given line. Assume an axis of revolution perpendicular to H to pass through the point a. Revolve the line about this axis until the H-projection takes the position $a^h c^h$, parallel to GL.

In this revolution, the angle which the line makes with the axis does not change. Consequently, the angle which the line makes with H, and therefore the length of the H-projection, remain constant. The point b revolves in the circular arc J (J^h, J^v), lying in a plane parallel to H (§ 75), and is found at the point c (c^h, c^v). The line in the position ac (a^hc^h, a^vc^v) is parallel to V, and a^vc^v shows the true length.

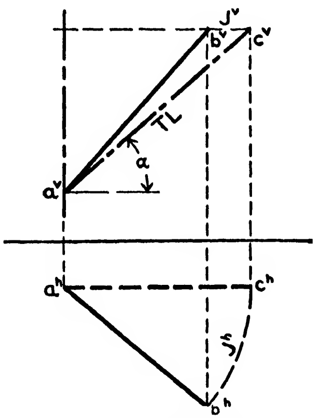

Fig. 100.

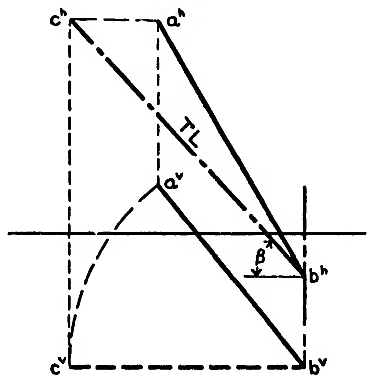

Fig. 101.

In a second example, Fig. 101, the axis of revolution is taken through the point b and perpendicular to V. The true length appears after the line has been revolved parallel to H.

79. The Angles Which a Line Makes with the Coördinate Planes. A line in the position ac (a^hc^h, a^vc^v), Fig. 100, which is parallel to V, shows in the V-projection the angle, α, which the line itself makes with H. The line ac is the revolved position of the line ab. In the preceding construction we saw that during the revolution the angle which the line made with H did not change. Hence α is the angle which the given line ab makes with H That is, in addition to finding the true length of the line ab in Fig. 100, we have incidentally found the angle which this line makes with H.

Similarly, in Fig. 101, we found, as an incidental part of the construction, the angle β which the line ab makes with V.

80. Second Method for Finding True Length. The true length of a straight line may also be found by revolving the line about one of its own projections as an axis, until the line lies in a coördinate plane.

Problem 3 (bis). ***To find the true length of a straight line.***

SECOND METHOD. By revolution about one of the projections of the line.

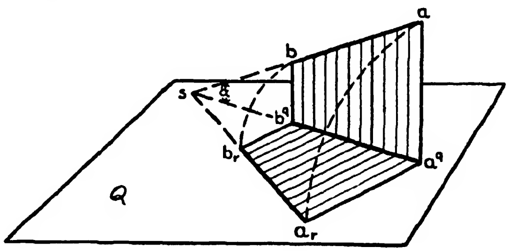

FIG. 102.

Analysis (Fig. 102). Let ab be the given line, projected on any plane Q at a^qb^q. Revolve the plane abb^qa^q about a^qb^q as axis into Q. Then a falls at a_r, where a_ra^q equals aa^q, and is perpendicular to a^qb^q (§ 76); b falls at b_r, with b_rb^q equal to bb^q and perpendicular to a^qb^q; hence a_rb_r equals ab, and shows the true length of the line.

Construction. The projection a^hb^h, Fig. 103, corresponds to a^qb^q, Fig. 102, and is taken as the axis of revolution. Since a^h is the projection of the point a in space, the distance, aa^h, of point a from the axis is equal to the distance of the point a from H, which shows in the V-projection as the distance from a^v to GL. Hence to find the revolved position a_r, make a^ha_r perpendicular to a^hb^h, and the distance a^ha_r equal to a^ve. Similarly, b^hb_r is perpendicular to a^hb^h, and is equal to b^vf. Then a_rb_r shows the true length of the line ab.

In Fig. 104 the V-projection is taken as the axis, and the line is revolved into V. The distances a^va_r and b^vb_r are the distances of the points a and b, respectively, from V, and are equal to the distances from a^h and b^h to the ground line. Note that not only the distances, but the directions, of the points a and b from the V-plane must be considered. The H-projection

shows that a and b, in space, are on opposite sides of V, hence their revolved positions, a_r and b_r, must fall on opposite sides of the axis, a^vb^v, which lies in V.

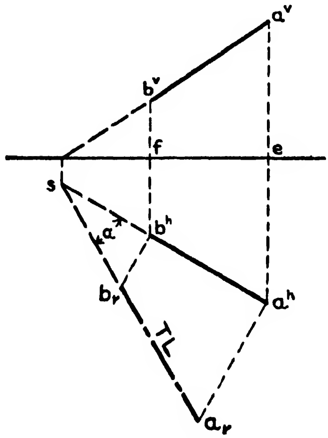

FIG. 103.

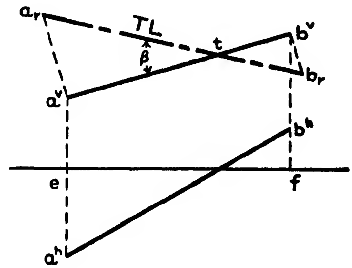

FIG. 104.

81. Traces of the Line. Let the line ab, Fig. 102, be produced to intersect its projection a^qb^q in the point s; then in revolving about a^qb^q as axis, the point s will remain fixed, and a_rb_r produced must pass through s. But since a^qb^q lies in Q, the point s must be the point in which ab pierces Q, or the Q-trace of ab. In the projection, Fig. 103, let a_rb_r be produced to meet a^hb^h produced at s; then s must be the H-trace of the line ab, and should be the same point that is found by projecting from the intersection of a^vb^v produced and GL, as in Problem 1 (§ 37).

82. Angles with the Coördinate Planes. The angle α, Fig. 102, between ab and a^qb^q is the angle which the line ab makes with the plane Q, and this angle is not changed by the revolution into Q. Hence, in Fig. 103, the angle α, between a^hb^h and a_rb_r, is the true angle which the line ab makes with H.

In Fig. 104, since the line ab was revolved into V, we have found the V-trace, t, where a_rb_r intersects a^vb^v. The angle, β, between these lines is the angle which the line ab makes with V.

83. Converse Problems. In each of the preceding methods for finding the true length of a line it is necessary to find the angle between the line and one of the coördinate planes. Conversely, it is easy to see that a line may be located, when certain angles are known, by reversing one of the preceding constructions.

Problem 4. ***To find the projections of a line of definite length, when its slope, the angle which it makes with one coördinate plane, and the direction of its projection on that plane, are known.***

In order to give the line a definite position in space, it will be assumed, in addition to the above data, that one end of the line is located at a given fixed point *a*.

It is evident geometrically that the position of the line in space is now completely determined.

Analysis. If the angle which the line makes with *H* is given, reverse any construction of Problem 3 which gives the angle with *H*. Proceed similarly if the angle with *V* is the given angle.

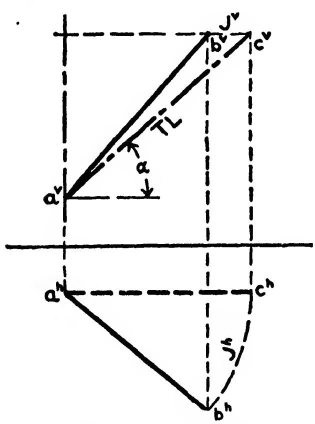

Fig. 100 (repeated).

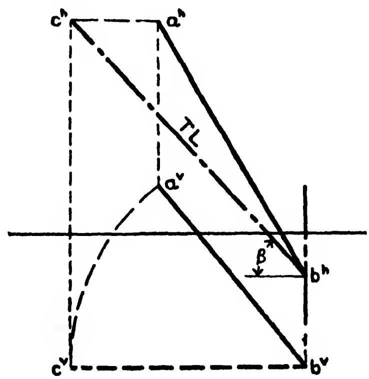

Fig. 101 (repeated).

Construction. First Method (Fig. 100). [Reverse of First Method, Prob. 3, § 78.] Let the angle with *H* and the direction of the *H*-projection be given. Let the point *a* be the fixed end of the line.

Place the line in the position ac (a^vc^v, a^hc^h), parallel to V, making a^vc^v equal to the true length of the line, at the given angle with H, and sloping upward or downward from a as given.

Take an axis perpendicular to H through the fixed end, a, and revolve the line until the H-projection takes the given direction a^hb^h. Find the corresponding projection a^vb^v.

SECOND METHOD (Fig. 103). [Reverse of Second Method, Prob. 3, § 80.] As before, let the angle, α, with H and the

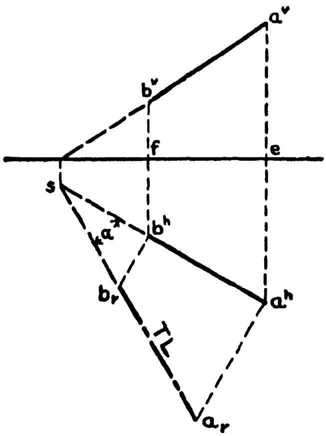

FIG. 103 (repeated).

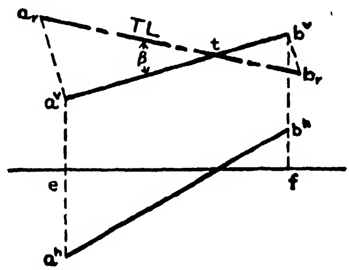

FIG. 104 (repeated).

direction of the H-projection be given, and let the point a be the fixed end of the line. Through a^h draw a^hs, indefinite in length, and in the given direction. Draw a^ha_r perpendicular to a^hs, and make the distance a^ha_r equal to a^ve. Draw a_rs indefinite in length and at the angle α with a^hs. In this case, the line is supposed to slope downward from the point a. Make a_rb_r equal to the given true length of the line. Draw b_rb^h perpendicular to a^hs, locating b^h. Project from b^h perpendicular to GL, and make the distance fb^v equal to b^hb_r, thus locating b^v.

The student should see if he can solve the problem when the angle with V and direction of the V-projection are given, by reversing the constructions of Figs. 101 and 104.

Problem 5. ***To find the projections of a line making given angles with H and V.***

In order to bring a definite solution it will be assumed that the true length, slope, and location of one end of the line are known. The general solution to the problem as stated above consists of four series of parallel lines, corresponding to the four possible slopes of a line (§ 31).

Analysis. Consider the problem solved, as in Fig. 105, and let ab (a^hb^h, a^vb^v) be the resulting line, with point a the fixed

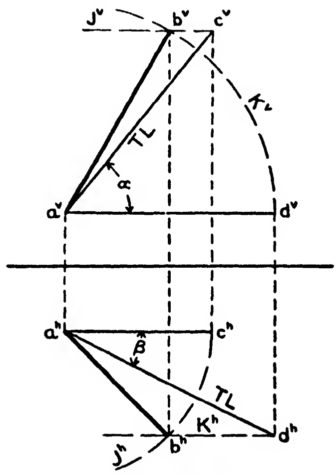

Fig. 105.

end. Take an axis through a perpendicular to H, and revolve the line to the position ac (a^hc^h, a^vc^v), parallel to V, where a^vc^v shows the true length of the line and its angle, α, with H (Prob. 3, First Method, § 78). Also take an axis through a perpendicular to V, and revolve the line to the position ad (a^vd^v, a^hd^h) parallel to H, where a^hd^h shows the true length of the line and the angle, β, with V. The problem may now be solved by reversing this construction.

Construction. Place the line in the position ac (a^vc^v, a^hc^h) parallel to V, where a^vc^v is equal to the true length of the line and makes with GL the angle, α, which the line makes with H.

Revolve this line about an axis through a perpendicular to H, by drawing the projections, J^h and J^v, of the arc in which point c revolves.

Place the line also in the position ad (a^hd^h, a^vd^v) parallel to H, where a^hd^h (equals a^vc^v) equals the true length of the line and makes with GL the angle, β, which the line makes with V. Revolve this line about an axis through a perpendicular to V, by drawing the projections, K^v and K^h, of the arc in which d revolves.

The point b (b^h, b^v), which determines the projections of the required line ab, is found at the intersection of the two paths of revolution J and K. Note that this point is independently determined in each projection; hence as a check on the work the projector b^vb^h should be perpendicular to GL.

CHAPTER IX

SOME SIMPLE INTERSECTIONS — DEVELOPMENTS

84. Simple Intersections of Solids of Revolution. In generating the surface of a solid of revolution, each point of the generating line describes a circle which lies in a plane perpendicular to the axis, and whose center lies in the axis (§ 74). If the axis of the surface be placed perpendicular to *H* (or *V*), these circles will project on *H* (*V*) in their true shape and size, and on *V* (*H*) as horizontal straight lines which are the edge views of the planes containing the circles (§ 75).

The intersection of a surface of revolution by a plane may be found by drawing on the surface a sufficient number of circles, obtained by passing auxiliary planes perpendicular to the axis of the surface, and then finding the points in which these circles are intersected by the given planes.

85. Visibility of the Intersection. In this and subsequent parts of this book, whenever a solid is intersected by a plane, no part of the solid will be considered as cut away or removed unless so stated. In determining the visibility of the line of intersection, the cutting plane will be considered transparent and the solid opaque. The line of intersection will be visualized as a line drawn on the surface of the solid. Hence, the line of intersection, or any portion of it, will be visible in any given projection when the line lies on a portion of the surface which is visible in that projection; and conversely.

The visibility of the line of intersection, then, depends directly upon the visibility of the given solid (§ 24).

86. Examples. The only position of the cutting plane which will be considered at this time is an edge view. Then

the plane is perpendicular to either H or V. In the following examples, the axis of each surface is perpendicular to H, so that the auxiliary circles (§ 84) all lie in planes which are parallel to H. Note in each example the visibility of the curve of intersection (§ 85).

EXAMPLE 1 (Fig. 106). *A cone of revolution cut by a plane perpendicular to V.* Since the whole convex surface of the cone is visible in plan, the entire curve is visible in plan.

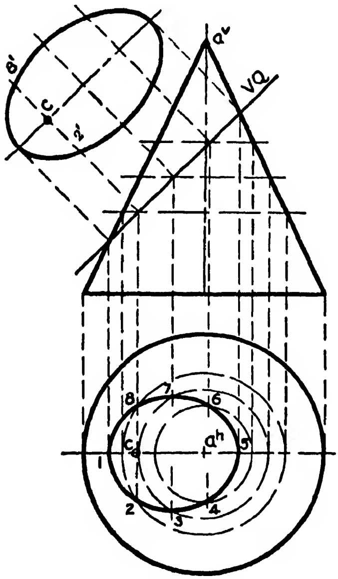

FIG. 106.

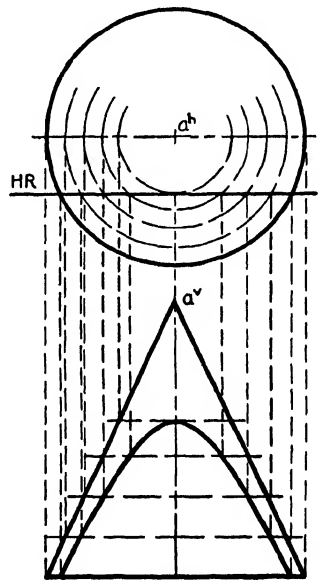

FIG. 107.

The true size of the section is found by revolving the section until parallel to V about an axis parallel to VQ. The true width at any point, as $2'8'$, equals the true distance 2–8 which appears in the H-projection.

EXAMPLE 2 (Fig. 107). *A cone of revolution cut by a plane parallel to V.* The curve is visible in elevation, since it is wholly on the front half of the cone.

The true size of the section appears in the V-projection.

EXAMPLE 3. (Fig. 108). *A sphere cut by a plane perpendicular to V.* Here the intersection, as seen in elevation, lies partly on the upper half of the sphere, and partly on the lower. Hence, in the plan one part of the curve is visible, and the other part invisible. The two points in which the curve changes from visible to invisible are necessarily on the outline of the sphere. Why?

The true size of the section is a circle whose diameter appears in the V-projection.

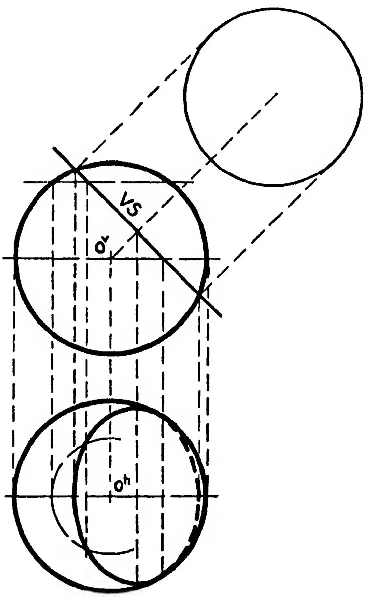

FIG. 108.

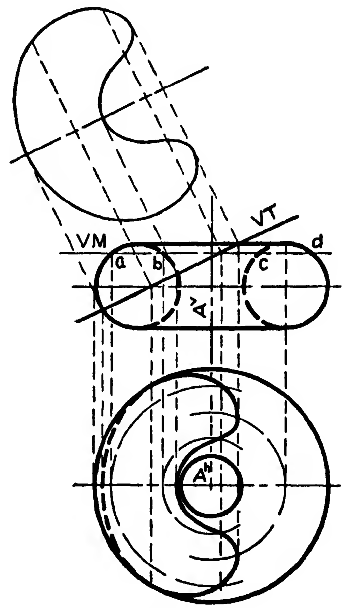

FIG. 109.

EXAMPLE 4 (Fig. 109). *A torus* (§ 26) *cut by a plane perpendicular to V.* An auxiliary plane, such as M, cuts the generating circle in two points. Hence, in this plane two circles lying in the surface of the torus can be drawn, one of diameter *ad*, and the other of diameter *bc*. Similarly, two circles can be drawn in every plane perpendicular to the axis, except in

those which pass through the highest and lowest points of the generating circles.

The visibility of the curve in plan is determined in the same way as for the sphere, Fig. 108.

The true size of the section is shown parallel to *V*, as in Example 1.

EXAMPLE 5 (Fig. 110). *A hyperbolic spindle, cut by a plane perpendicular to V.* In this case the surface that is cut is wholly hidden in plan, hence the curve is invisible.

The true size of the section is shown parallel to *V*.

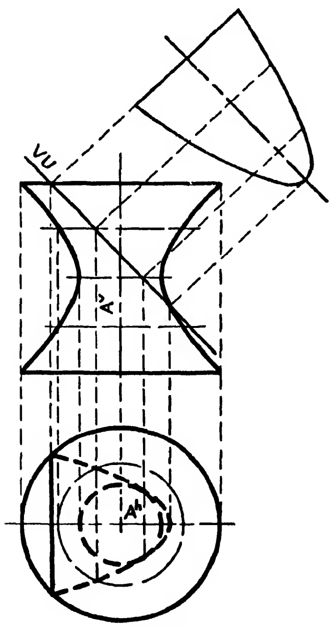

FIG. 110.

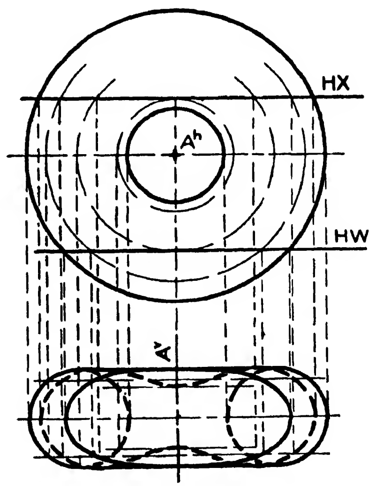

FIG. 111.

EXAMPLE 6 (Fig. 111). *A torus, cut by two planes parallel to V.* The *X* section, wholly on the back of the surface, is invisible. The *W* section, entirely on the visible part of the torus, is wholly visible.

The true size of each section appears at once.

EXAMPLE 7 (Fig. 112). *A hyperbolic spindle, cut by two planes parallel to V.* Each plane cuts a section composed of

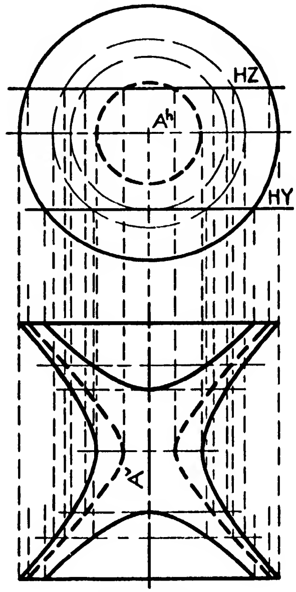

FIG. 112.

two parts. The visibility of the curves should be evident from an inspection of the position of the cutting planes.

87. Developments. The **development** of a solid bounded by plane faces consists of a series of polygons of the true sizes and shapes of the various faces of the solid, so arranged, edge to edge, that the development may be folded up to reproduce the surface of the solid. If a face of a solid is a triangle, the true size of the face may be constructed from the true lengths of its three sides. If the face is a polygon of more than three sides, the true size and shape cannot be determined from the true lengths of the sides alone. Any plane polygon, however, can be divided into triangles by drawing certain diagonals, and the polygon can be constructed from these triangles. Hence the development of any solid bounded by plane faces can be obtained wholly by finding the true lengths of straight lines.

88. Working Method for Finding the True Length of a Line. In the solution of many problems the true length of a line is found in order to be used as a radius for setting compasses or dividers. The angle made with H or V is not needed. Hence it is sufficient to construct two points whose distance apart is equal to the true length of the line.

Let ab, Fig. 113, be any line, and let the true length of this line be found by revolving it parallel to V, keeping the upper

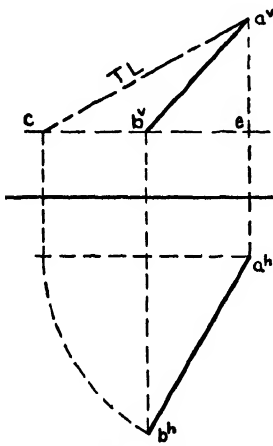

Fig. 113.

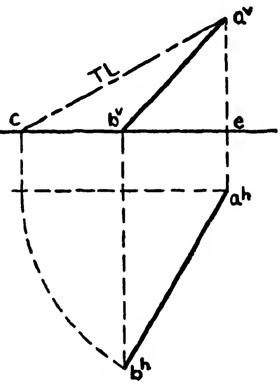

Fig. 114.

end of the line fixed (Prob. 3, First Method, § 78), thus giving a^vc as the true length. Now the point c can be found by drawing through b^v a horizontal line, intersecting the projector a^va^h at e, and then laying off the distance ec directly with the dividers equal to a^hb^h. Hence the following working rule:

Rule. *The true length of a straight line equals the hypothenuse of a right triangle whose base equals the length of the H-projection of the line, and whose altitude is equal to the difference in elevation between the two ends of the line.*

If the lower end of the line is in the H-plane, b^v will lie in GL, and the method becomes even simpler. Thus, the true length of the line ab, Fig. 114, may be found by laying off on GL the distance ec equal to a^hb^h; then a^vc is the desired true length.

89. To Find the Development of a Solid with Plane Faces. Let the given solid be the frustum of an irregular four-sided pyramid, Fig. 115. Each face is a quadrilateral, which can be divided into two triangles by means of one diagonal. Nevertheless it is easier to begin by finding the development of the triangular faces of the complete pyramid. Assuming that the true lengths of the various lines are found as needed, we proceed with them as follows (*B*, Fig. 115). On any convenient line,

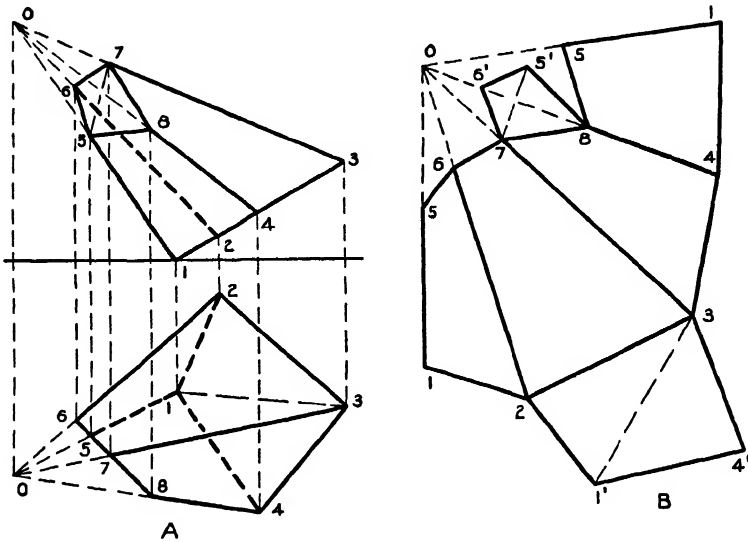

Fig. 115.

lay off the distance 0–1; with 0 and 1 as centers, and radii 0–2 and 1–2 respectively, strike arcs intersecting in point 2. This gives the development of the triangular face 0–1–2.

With 0 and 2 as centers, radii 0–3 and 2–3 respectively, locate point 3, thus obtaining the development of the face 0–2–3. In the same way obtain the faces 0–3–4 and 0–4–1.

The base, 1–2–3–4, of the pyramid is divided into triangles by the diagonal 1–3, and is plotted to join one of the edges previously located, as 2–3. The result thus far obtained is the development of the complete pyramid. To convert it into the development of the frustum, proceed as follows :

On 0–1, 0–2, etc., of the development, measure the true lengths of the lines 1–5, 2–6, 3–7, 4–8, and draw 5–6–7–8–5.

Draw one diagonal, as 5–7, of the upper base, plot this base by means of the two triangles thus formed, and join it properly to one of the previous lines of the development. The development of the frustum is now complete.

90. To Construct the Projections of a Prism Whose Long Edges Make Given Angles with *H* and *V* (Fig. 116). Let it be required to draw the projections of a prism whose long edges shall make 30° with *H* and 45° with *V*.

Place the prism first with its long edges parallel to *H* and making the given angle, 45°, with *V*. Assuming that the necessary sizes, slopes, and distances are given, we can construct

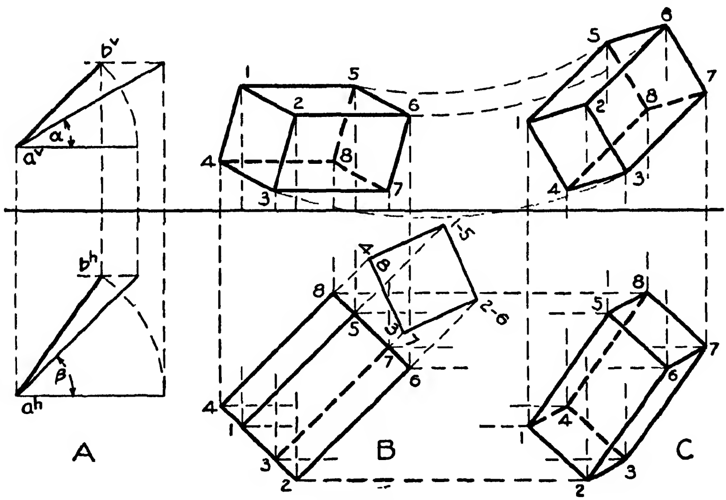

FIG. 116.

these projections by using a secondary ground line, as in § 72 (*B*, Fig. 116). Find, by Problem 5, § 83, the projections of a line *ab* making the given angles with *H* and *V* (*A*, Fig. 116).

Now the long edges of the prism as placed in *B*, Fig. 116, make the given angle with *V*. If the prism be revolved about any axis perpendicular to *V*, the edges will continue to make the same angle with *V*, while the projection of the prism on *V*

will remain of the same shape and size. Let the prism be revolved until the V-projections of the long edges are parallel to $a^v b^v$, as shown at C, Fig. 116.

In making this revolution, the actual axis need not be used. It suffices to copy the V-projection obtained at B, making the long edges of the prism parallel to the direction $a^v b^v$ obtained at A. But in this revolution, the H-projection of the path of each moving point will be a horizontal straight line, regardless

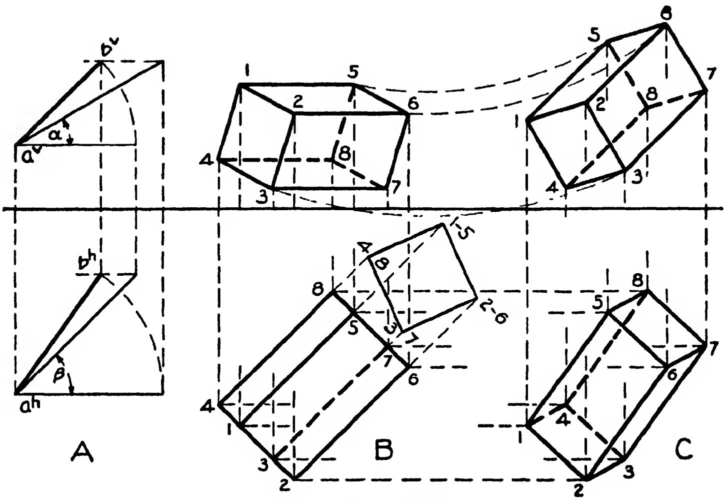

FIG. 116 (repeated).

of the position of the axis of revolution. Hence the H-projection at C may be obtained by projecting vertically from the V-projection, and horizontally from the H-projection at B.

As a check, the long edges of the projection thus obtained should be parallel to the direction $a^h b^h$ found at A.

The projections at C are the required projections.

91. Other Methods. The projections at B, Fig. 116, were made by knowing the angle which the long edges of the prism made with V, while the projections at C were made by knowing the inclination of the V-projection of the edges. These two quantities might have been given directly, and projections B and C constructed by Problem 4, § 83.

CHAPTER X

LINES IN A PLANE — PARALLEL LINES AND PLANES

92. Intersecting and Parallel Lines. Two lines intersect when they pass through a common point. Two lines are parallel when they are everywhere equally distant. If two straight lines lie in the same plane, they either intersect or are parallel. This is not necessarily the case, however, with any two lines in space.

93. Parallel Lines in Space. *If two lines in space are parallel, their projections on any plane are parallel.*

Let the two parallel lines A and B, Fig. 117, be projected on the plane Q by planes perpendicular to Q. Since the two

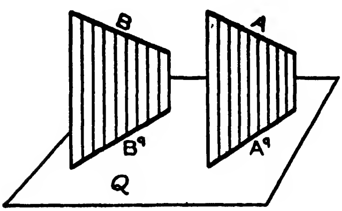

FIG. 117.

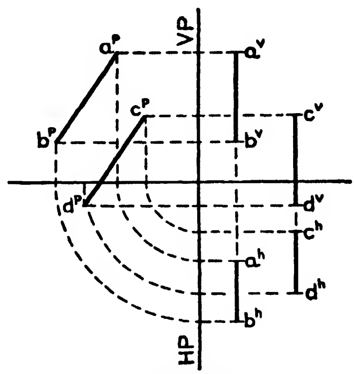

FIG. 118.

projecting planes will be parallel, their intersections, A^q and B^q, with the plane Q will be parallel.

In general, if lines A and B are parallel, it is definitely shown by the simultaneous condition that A^h is parallel to B^h, and A^v is parallel to B^v. Profile lines are an exception; such lines may be tested by means of their profile projections. Thus, the profile lines ab and cd, Fig. 118, are parallel, since their profile projections are parallel.

94. Intersecting Lines in Space. *If two lines in space intersect, their corresponding projections intersect in the same projector.*

Let A and B, Fig. 119, be two intersecting lines. Since these lines intersect, there must be a point, c, common to each of them. Since c is a point of the line A, the H-projection, c^h,

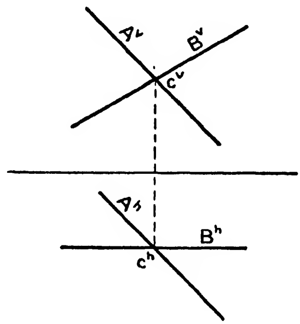

FIG. 119.

must lie on A^h (§ 36). Moreover, since c is in the line B, c^h must lie on B^h. Hence c^h is at the intersection of A^h and B^h.

Similarly, the V-projection of c, c^v, must lie at the intersection of A^v and B^v. But since c^h and c^v are two projections of the same point, they must lie in the same projector.

If two lines have a common projection (Fig. 120), they lie in the same plane, namely, the plane whose edge view coincides

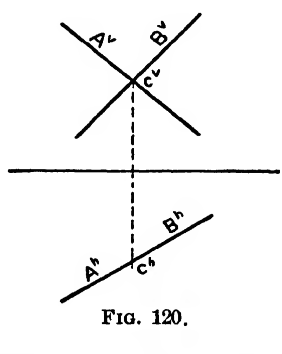

FIG. 120.

with the common projection. Such lines are either intersecting, as in the figure, or parallel (§ 92).

95. Test for Intersecting Lines. Conversely, *if the corresponding projections of two lines intersect in the same projector, the lines in space intersect.* The only exception occurs when one of

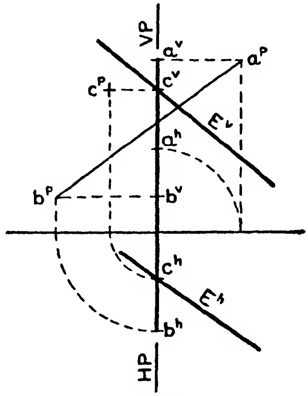

FIG. 121.

the lines is a profile line; thus, the line E, Fig. 121, does *not* intersect the profile line ab, since the point c, in which E pierces the profile plane containing ab, does not lie on the line ab.

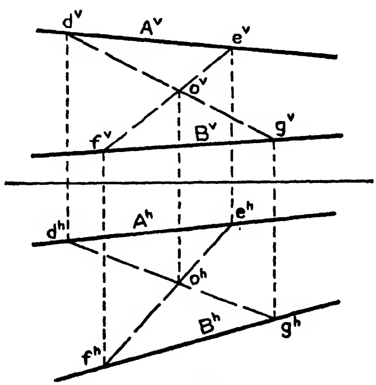

FIG. 122.

If the point of intersection is outside the limits of the drawing (Fig. 122), the test fails. In this case take any two points, d and e, on A, and any two, f and g, on B. Then if A and B intersect, df and eg (or else dg and ef) will intersect.

96. A Line in a Plane. *If a line lies in a plane, the traces of the line lie in the corresponding traces of the plane.*

Let a line A lie in a plane Q (Figs. 123 and 124). Then if A intersects H, it can evidently do so only in some point of

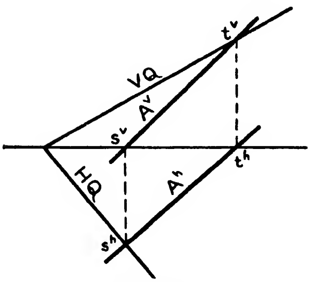

FIG. 123.

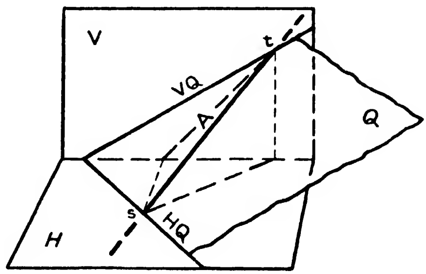

FIG. 124.

the line in which Q intersects H; and the intersections with H are the respective traces of the line and plane. Similarly for any other plane of projection.

97. To Locate a Line in a Plane. A line will lie in a given plane if the line passes through two points which lie in the plane. Thus, in Fig. 123, let us assume s^h on HQ, and t^v on VQ. These two points are the traces of some line lying in the plane Q (§ 96). The projections A^h and A^v may then be found by Problem 2, § 37.

Other methods of determining a line so that it shall lie in a given plane will be given later. (See §§ 100, 133.)

98. A Plane Containing a Given Line. The proposition of § 96 enables us to pass a plane through a given line. Thus, in Fig. 123, let $A(A^h, A^v)$ be a given line, whose traces are the points s and t. A plane Q may be passed through A by drawing HQ in any direction through s^h, and then drawing VQ through t^v and the point in which HQ intersects the ground line. Conversely, VQ may be drawn first, in any direction through t^v, and then HQ may be drawn through s^h and the intersection of VQ with the ground line.

An indefinite number of planes may be passed through a

given line. In particular (Fig. 125), we may draw a plane X, perpendicular to H, a plane Y perpendicular to V, and a plane S parallel to the ground line. (See also § 102.)

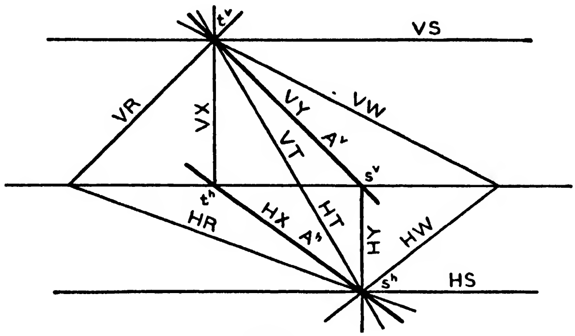

FIG. 125.

99. Principal Lines of a Plane. For every plane there are two systems of lines which are of importance, and are called the **principal lines** of the plane. One of these systems is parallel to H, the other parallel to V.

The horizontal principal lines are H-parallels, cut from the given plane by a series of planes parallel to H. The lines are therefore parallel to the H-trace of the plane; their H-projections are parallel to this trace, while their V-projections are parallel to the ground line. Visualize this statement, aided, if necessary, by Fig. 126.

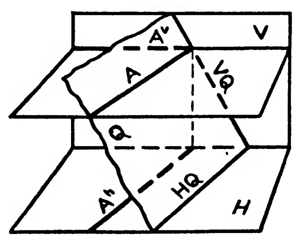

FIG. 126.

The vertical principal lines are cut from the plane by a series of planes parallel to V. They are V-parallels, parallel to the V-trace of the plane. Their V-projections are parallel to this trace, their H-projections are parallel to the ground line.

These two groups of lines are, in general, different. They reduce to the same set when, and only when, the given plane is parallel to the ground line.

100. To Project the Principal Lines of a General Plane. To draw the projections of a horizontal principal line of a general plane, Q, Fig. 127, assume the V-trace of this line as any point,

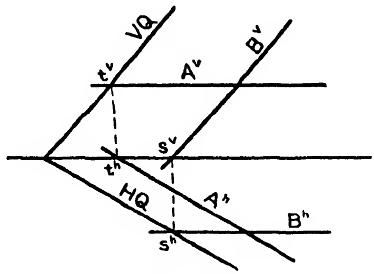

FIG. 127.

t^v, in VQ, and find t^h on GL. Then, by § 99, A^h passes through t^h and is parallel to HQ, while A^v passes through t^v and is parallel to GL.

Similarly, a vertical principal line, B, may be drawn by first assuming its H-trace, s, on HQ.

101. To Project the Principal Lines of a Plane Parallel to the Ground Line. To draw a principal line of a plane when the plane is parallel to the ground line, as Q, Fig. 128, find the

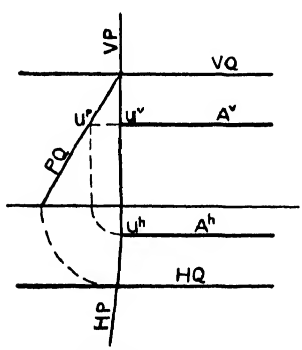

FIG. 128.

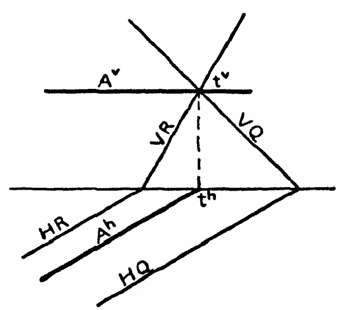

FIG. 129.

profile trace, PQ, of the given plane. On PQ assume any point, u^p, as the profile trace of the required principal line. Both projections of this line are parallel to the ground line.

102. A Plane Containing a Line Parallel to *H* or *V*. Any plane, as *Q* or *R*, Fig. 129, which is passed through an *H*-parallel will have its *H*-trace parallel to the *H*-projection of the line. (Converse of § 100.)

Similarly, any plane passed through a *V*-parallel will have its *V*-trace parallel to the *V*-projection of the line.

Any plane, as *Q*, Fig. 128, which contains a line parallel to both *H* and *V* will have both its *H*- and *V*-traces parallel to the ground line. (Converse of § 101.)

103. Parallel Planes. *If two planes are parallel, their corresponding traces are parallel.*

Let *Q* and *R* be the given planes. Let *Q* intersect *H* in *HQ*, and *R* intersect *H* in *HR*. The figure is left for the student to draw. Now when two parallel planes, *Q* and *R*, are intersected by a third plane, *H*, the lines of intersection must be parallel. Hence *HQ* is parallel to *HR*. Similarly for any other plane of projection.

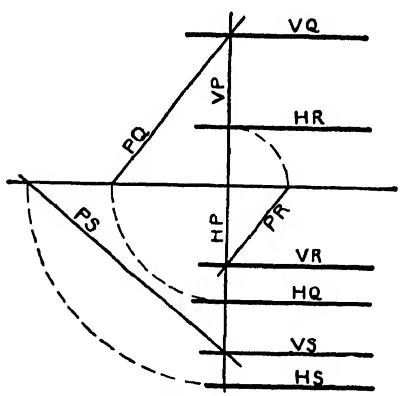

FIG. 130.

Conversely, if the *H*- and *V*-traces of two planes are parallel, the planes are, in general, parallel. The only exception occurs with planes parallel to the ground line. Such planes may be tested by means of their profile traces. Thus, in Fig. 130, the plane *R* is parallel to the plane *Q*, while the plane *S* is not.

104. A Line Parallel to a Plane. A straight line is parallel to a plane if it is parallel to some line lying in the plane. Thus the line A, Fig. 131, is parallel to the plane Q, since it is parallel to the line B which lies in Q. If the line B were not

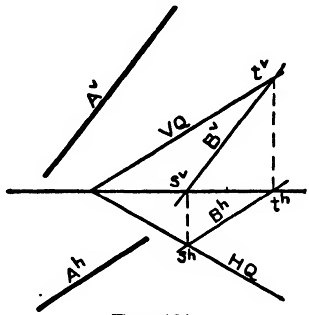

FIG. 131.

drawn, however, it would be difficult to recognize the parallelism. Thus, in Fig. 132, both the lines A and B are parallel to the plane R, but this fact can hardly be seen by inspection.

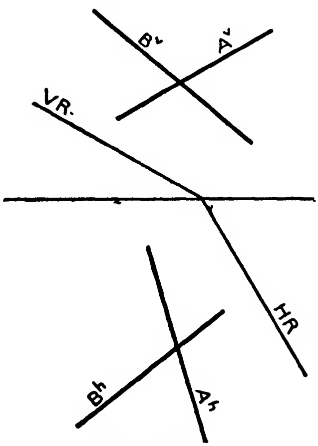

FIG. 132.

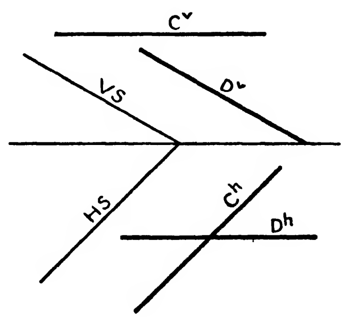

FIG. 133.

The fact that a line is parallel to a plane can be recognized at once only when the line is parallel to one of the systems of principal lines of the plane. In Fig. 133, it is evident that both lines C and D are parallel to the plane S.

105. A Plane Parallel to a Line. Conversely, a plane is parallel to a straight line if the plane contains a line which is parallel to the given line. Thus, in Fig. 131, since the lines A and B are parallel, the plane Q, or any other plane passed through B, is parallel to the line A. The only exception is the plane which contains both A and B.

106. A Plane Determined by Lines or Points. One plane, and one only, may be found which contains

(*a*) two intersecting lines;
(*b*) two parallel lines;
(*c*) a line and a point not on the line;
(*d*) three points not in the same straight line.

Problem 6. ***To find the plane which contains two given intersecting or parallel lines.***

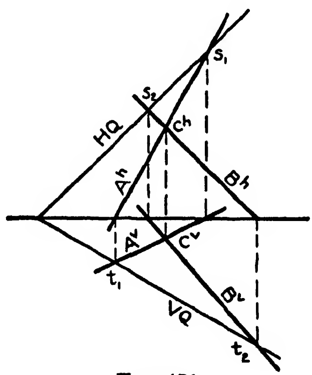

FIG. 134.

Analysis. The trace of the required plane on any coördinate plane must contain the trace of each of the given lines (§ 96). The plane is found when its H and V traces are found.

Construction (Fig. 134). Let A and B, intersecting at point c, be the given lines. Find the traces of A and B (Prob. 1, § 37). The required trace HQ is determined by the two H-traces s_1 and s_2; the trace VQ by the two V-traces t_1 and t_2.

Check. The H- and V-traces of a plane must intersect on the ground line (§ 46). Hence HQ and VQ, when produced, must intersect on GL.

A plane which contains two given parallel lines is shown in Fig. 135. The construction is entirely similar to that just given.

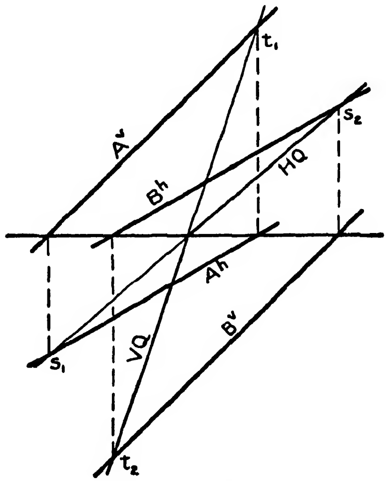

Fig. 135.

In Fig. 136 the *H*-trace of the given line *B* is too far removed to be used in the construction. The plane *Q*, containing *A* and

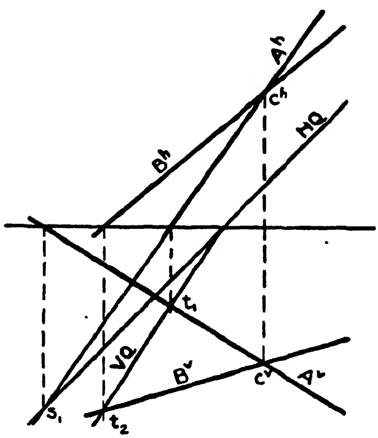

Fig. 136.

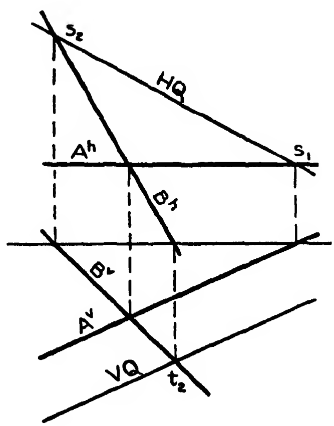

Fig. 137.

B, is located by first drawing *VQ* through the *V*-traces of the given lines; then *HQ* is drawn through the *H*-trace of *A* and

the point in which VQ intersects the ground line. This construction gives no check on the work, but may be used in case of necessity. Other constructions which may be employed when traces of the given lines are not accessible will be given in § 108. We shall consider at present only those cases in which one or both of the given lines are parallel to H or to V.

Special Case I. Let us suppose that one of the given lines is parallel to H or V. Let A, Fig. 137, be parallel to V. Then VQ is parallel to A^v (§ 102), and passes through the V-trace of the line B. The trace HQ is determined, as in the general case, by the H-traces of the given lines. In Fig. 137, the point in which HQ and VQ intersect on GL is not available, but it is not necessary.

Special Case II. If one of the given lines, as A, Fig. 138, is parallel to both H and V, that is, parallel to the ground line,

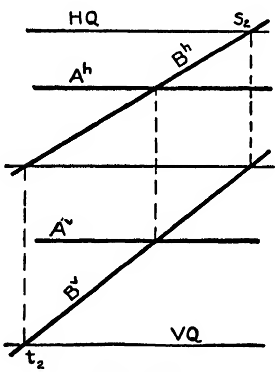

Fig. 138.

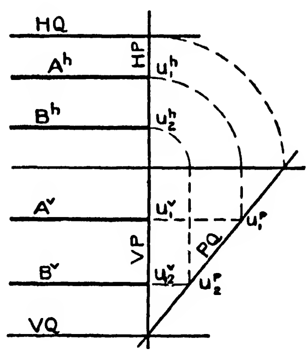

Fig. 139.

both HQ and VQ are parallel to the ground line (§ 102), one point on each trace being located by the line B.

Special Case III. Suppose both given lines parallel to the ground line. Let A and B, Fig. 139, be the given lines; then HQ and VQ are both parallel to GL. To find a point on each, we may find first the profile trace of the plane Q on any assumed profile plane; this trace, PQ, will pass through the profile traces u_1^p and u_2^p, of A and B respectively.

COROLLARY I. *To find the plane which contains a given line and a given point.*

Analysis. Through the given point, draw an auxiliary line parallel to the given line, thus reducing the problem to one of two parallel lines. Or, assume any point on the given line, and draw an auxiliary line through this point and the given point, thus reducing the problem to two intersecting lines. The first method is the one usually adopted.

Construction (Fig. 140). A is the given line, c the given point. The auxiliary line B is drawn through c parallel to A

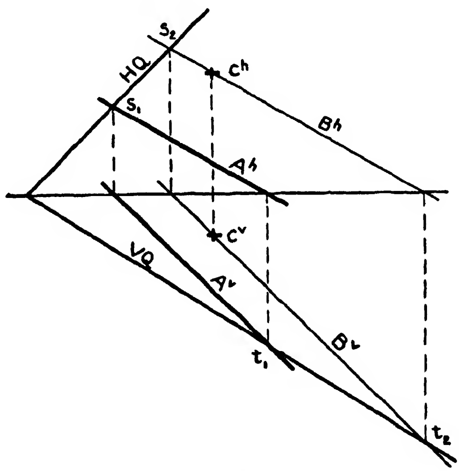

FIG. 140.

(§ 93), and the required plane Q is passed through the lines A and B.

In Fig. 141 the line A is parallel to V; hence the direction of VQ is known to be parallel to A^v (§ 102). But as neither A nor B has a V-trace, a point on VQ is determined by noting where HQ intersects GL.

COROLLARY II. *To find the plane which contains three given points not in the same straight line.*

Analysis. Connect any two points by a straight line, and through the third point draw a second line parallel to the first,

thus reducing the problem to two parallel lines. Or, draw lines connecting any point with each of the other two, thus reducing the problem to one of two intersecting lines.

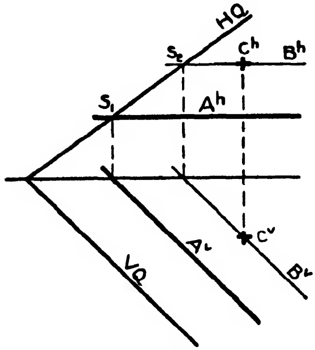

FIG. 141.

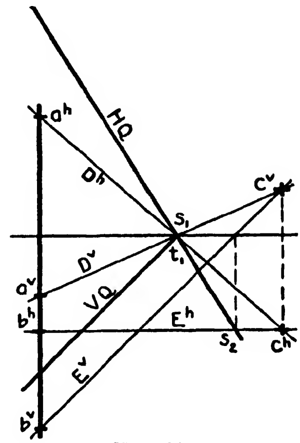

FIG. 142.

No figure for the general case is thought necessary. It is interesting to note that this device may be used for passing a plane through a profile line and a point. Thus (Fig. 142), let *ab* be the given line and *c* the given point. Draw the lines *ac* ($a^h c^h$, $a^v c^v$), and *bc* ($b^h c^h$, $b^v c^v$), and pass the required plane *Q* through these lines. This avoids the necessity of finding the traces of the profile line *ab*, although its traces, if found, should lie on *HQ* and *VQ* respectively (§ 96).

107. A Plane Parallel to Lines or to Another Plane. A plane which is parallel to given lines or to another plane is determined by the fact that a plane containing one of two parallel lines is parallel to the other line (§ 105).

One plane, and one only, may be found which

(*a*) contains a given line and is parallel to a second given line that is not parallel to the first line;

(*b*) contains a given point and is parallel to each of two given lines that are not parallel to each other;

(*c*) contains a given point and is parallel to a given plane.

Problem 7. *To find the plane which contains a given line and is parallel to a second given line.*

Analysis. Through any point on the first given line draw an auxiliary line parallel to the second given line. The required plane is determined by the first line and the auxiliary line.

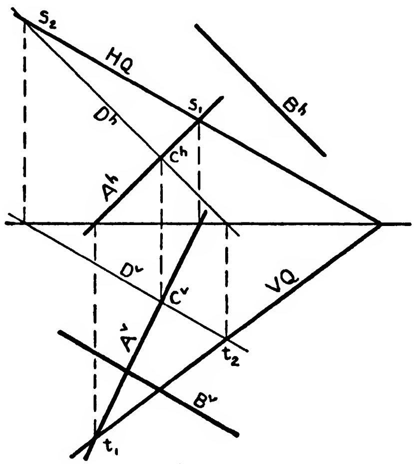

FIG. 143.

Construction (Fig. 143). Let it be required to pass the plane through the line A. Assume any point c on A; through c draw the line D parallel to B. Pass the required plane Q through the lines A and D (Prob. 6, § 106).

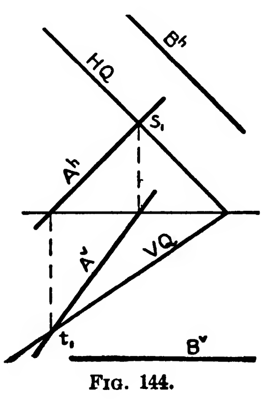

FIG. 144.

Special Case I. Suppose the second line is parallel to *H* or *V*; then no auxiliary line is needed. Thus, in Fig. 144, let us find the plane which contains the line *A* and is parallel to *B*. The *H*-trace of any plane which is parallel to *B* must be parallel to B^h (§ 104). Hence, find the traces of *A*. Draw *HQ* through s_1 parallel to B^h; draw *VQ* through t_1 and the point in which *HQ* intersects *GL*.

Special Case II. Suppose that either the first or the second given line is a profile line. The general solution will apply to this case; but if the problem be solved in this manner, a profile projection will be necessary. A simple construction, by which the use of a profile projection may be avoided, is as follows:

Let *ab* and *C*, Fig. 145, be the given lines. Through one point of the profile line, as *a*, draw an auxiliary line *D* parallel to the line *C*; through the other point, *b*, of *ab*, draw a second

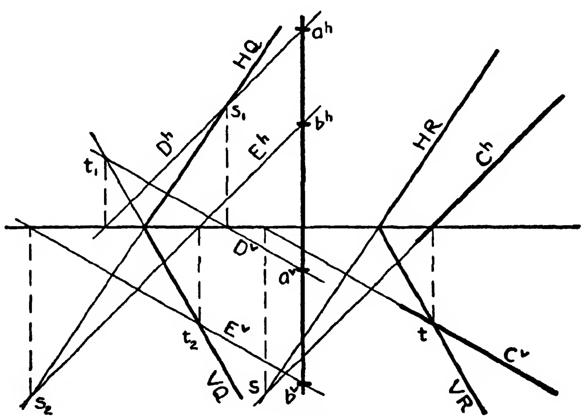

Fig. 145.

auxiliary line *E* parallel to *C*. Pass the plane *Q* through the two parallel lines *D* and *E*. Then if the required plane is to contain *ab* and be parallel to *C*, plane *Q* is the required plane. If, however, the required plane is to contain *C* and be parallel to *ab*, the plane *R*, passed through *C* parallel to plane *Q* (§ 103), is the required plane.

Problem 8. ***To find the plane which contains a given point and is parallel to each of two given lines.***

Analysis. Through the given point draw two auxiliary lines, one parallel to one given line, the other parallel to the other given line. The required plane is determined by the two auxiliary lines.

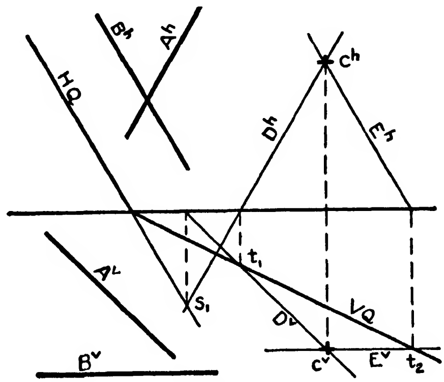

Fig. 146.

Construction (Fig. 146). Let c be the given point, A and B the given lines. Through c draw the auxiliary lines, D parallel to A, and E parallel to B. Pass the required plane Q through the lines D and E (Prob. 6, § 106).

The special cases of this problem are too similar to those of Problem 7 to require a detailed discussion.

Problem 9. ***To find the plane which contains a given point and is parallel to a given plane.***

Analysis. Through the given point, pass a line parallel to the given plane. Through this line, pass the required plane parallel to the given plane.

Construction (Fig. 147). Let Q be the given plane and a the given point. A line may be drawn through a and parallel to Q by drawing it parallel to either the horizontal or vertical principal lines of Q (§ 104). The line M, drawn through a, is parallel to the vertical principal lines of Q; through M pass the required plane T, parallel to Q (§ 103).

In Fig. 148, the required plane T is found by means of the auxiliary line N, which is parallel to the horizontal principal lines of the given plane Q.

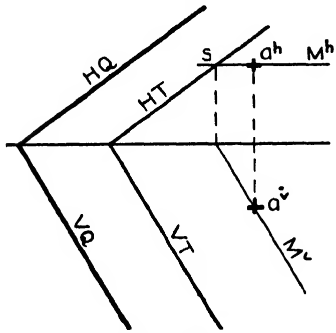

FIG. 147.

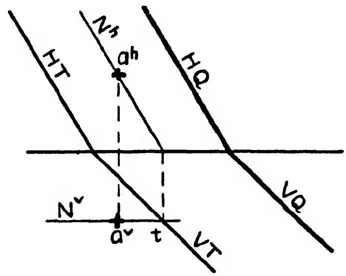

FIG. 148.

SPECIAL CASE (Fig. 149). Let the given plane Q be parallel to the ground line. Then the auxiliary line M, drawn through a and parallel to the principal lines of Q, is parallel to GL (§ 101) and has no traces on H and V. Pass the auxiliary

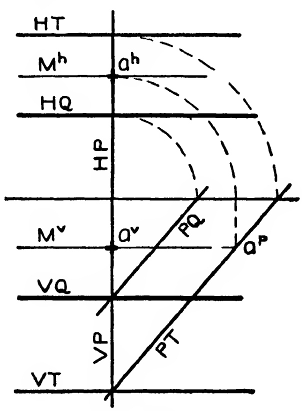

FIG. 149.

profile plane P through the given point. Find the profile trace, PQ, of Q, and the profile projection, a^p, of a. The profile trace, PT, of the required plane will pass through a^p and be parallel to PQ (§ 103). From PT may be found HT and VT.

108. Use of Auxiliary Lines in Finding the Traces of Planes. A straight line is determined:

1. When two of its points are known;
2. When one of its points and its direction (such as parallel to another line) are known.

For accuracy of construction, the second method of determining a line is often better than the first. In any event, however, a line is not determined until at least one point is known.

The traces of the required planes in the preceding problems, being straight lines, have necessarily been located by one of these methods. It occasionally happens, however, that necessary points for locating these traces fall far outside any reasonable limits for the size of the figure, and recourse must be had to auxiliary lines which will locate points within reach.

EXAMPLE 1 (Fig. 150). Let it be required to find the plane which contains the lines A and B, so situated that only one

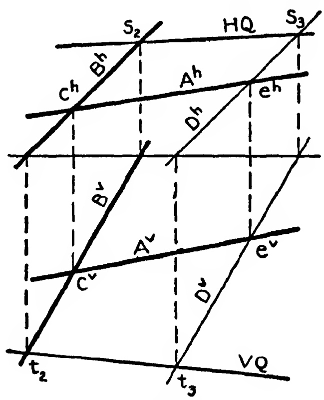

FIG. 150.

point, s_2, on HQ, and one point, t_2, on VQ, can be found. Assume any point, e, in the given line A; through this point draw the line D, parallel to the given line B. Pass the plane Q through the parallel lines B and D. This plane must necessarily contain the line A, since A intersects both B and D, and is the required plane.

In general, assume any point on one of the given lines.

Through this point draw an auxiliary line parallel to the second given line; the auxiliary line will lie in the plane of the given lines. Judgment must be used in selecting the line to which the auxiliary line should be made parallel, in order that this latter line may be of service in obtaining additional points.

EXAMPLE 2. Another method of obtaining a line lying in the plane of two given lines is shown in Fig. 151. Let *A* and *B* be the given lines. Assume any point, *e*, in *A*, and any point, *f*, in *B*. Then the line *D*, passing through *e* and *f*, will lie in the plane of the lines *A* and *B*, and the traces of *D* will lie in the traces of the required plane *Q*. As before, judgment must be used in selecting the points *e* and *f*.

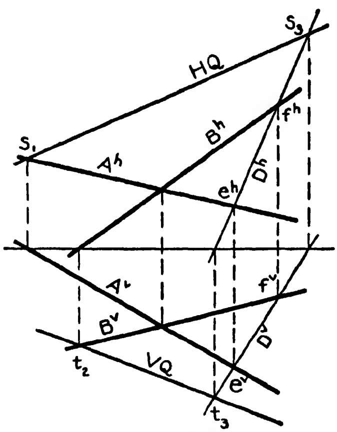

FIG. 151.

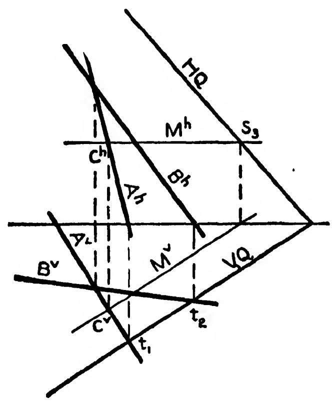

FIG. 152.

EXAMPLE 3. When the given data are sufficient to determine one trace of the plane, a very useful auxiliary line is one of the principal lines (§ 99) of the plane itself. Let *A* and *B*, Fig. 152, be sufficient to locate *VQ* but not *HQ*. Assume a point in one of the given lines, as point *c* in line *A*. Through *c* draw a vertical principal line of *Q*, by making M^v parallel to *VQ*, and M^h parallel to *GL*. Then the line *M* lies in *Q*, and its *H*-trace, s_3, is a point in *HQ*.

If *HQ* were the known trace, a horizontal principal line of the plane could be similarly drawn to locate a point in *VQ*.

Example 4. Another method of procedure, when one trace of the plane is known, is shown in Fig. 153. Let VQ be the known trace. Assume any point, t^v, in VQ; this point lies in V, and t^h is in GL. Through the point t, draw the auxiliary line D parallel to one of the given lines, as A. Then the line D lies in the plane Q, and its H-trace, s^h, is a point in HQ.

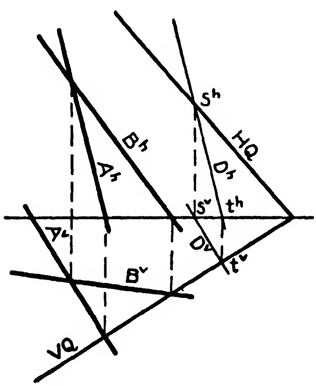

Fig. 153.

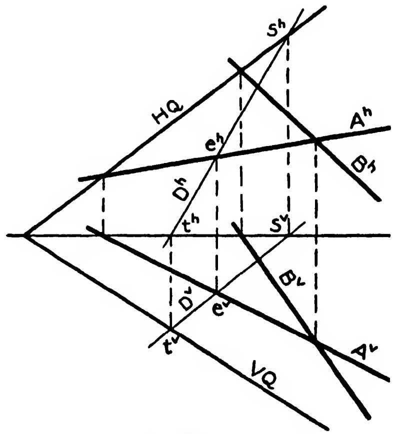

Fig. 154.

In general, the auxiliary line D may be drawn parallel to either of the given lines A or B; but in this case D should obviously be drawn parallel to A in order to obtain a result within the limits of the figure.

Example 5. Let the lines A and B, Fig. 154, be sufficient to locate HQ but not VQ. A third method of obtaining an auxiliary line is to assume any point, as s, on HQ, and any point, as e, on either of the given lines A or B. Then the line D, joining s and e, lies in the plane Q, and its V-trace, t^v, is a point in VQ.

In some situations, in order to get a sufficient number of points, it may be necessary to employ more than one auxiliary line. This would have been the case in Figs. 152, 153, and 154, if in any one of these figures the intersection of HQ and VQ with the ground line had not been available.

CHAPTER XI

PERPENDICULAR LINES AND PLANES

109. Perpendicular Lines. If two straight lines are perpendicular to each other, the fact is not, in general, apparent from the projections of the lines. For example, consider the square prism represented in Fig. 94. The edges of this solid form

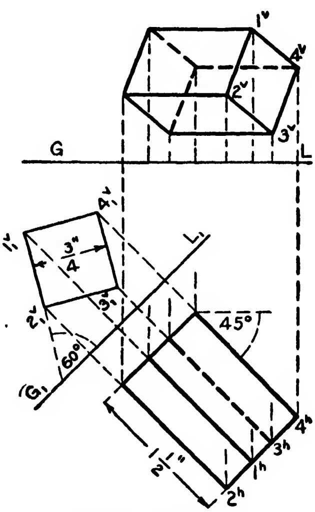

FIG. 94 (repeated).

three series of mutually perpendicular lines; but all these edges are oblique to V, and in the V-projection no right angles appear.

In the H- and secondary V-projections, however, the projected edges are at right angles. This shows that, under certain conditions, it is possible to recognize perpendicular lines from their projections.

110. Perpendicular Lines Whose Projections Are Perpendicular. *If two lines in space are mutually perpendicular, and are projected on any plane of projection parallel to one of the lines, the two projections will be perpendicular to each other.*

There is one and only one exception. The plane of projection must not be perpendicular to the second line; for in this case the line would project as a point, and a point cannot be said to be perpendicular to a line.

Let the line A, Fig. 155, be parallel to H. Let Q be a plane perpendicular to A and consequently perpendicular to H. In this position of the line and plane, it is evident that A^h and HQ are perpendicular to each other. Now let B be any line drawn in the plane Q. Since A is perpendicular to Q, the lines A and B must be mutually perpendicular. Also, since

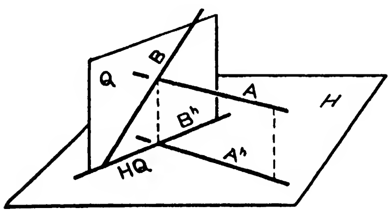

FIG. 155.

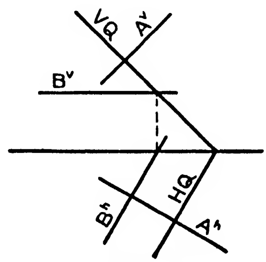

FIG. 156.

Q is perpendicular to H, B^h must coincide with HQ; that is, the projections A^h and B^h are perpendicular to each other. It may be noted that the line B need not necessarily intersect the line A, since any line lying in Q is considered to be perpendicular to A.

What is true of the H-plane of projection must be true of any plane of projection, hence the proposition is established.

111. A Line Perpendicular to a Plane. *If a line is perpendicular to a plane, any projection of the line is perpendicular to the corresponding trace of the plane.*

Let the line A, Fig. 156, be perpendicular to the plane Q. Then A will be perpendicular to any line lying in Q, in particular to the horizontal principal line B. But B is parallel to

H; hence A^h is perpendicular to B^h, by § 110. Moreover B^h is parallel to HQ. Hence A^h is perpendicular to HQ. Similarly for any other plane of projection.

112. Test for Perpendicularity of a Line and a Plane. Conversely, *if the horizontal and vertical projections of a line are perpendicular to the corresponding traces of a plane, the line and plane are mutually perpendicular.*

The only exception occurs when the plane is parallel to the ground line, for the projections of any profile line on H and V are perpendicular to the H- and V-traces of such a plane. Such cases may be tested by the use of the profile projection. Thus, in Fig. 157, the line ab is found to be perpendicular to the plane Q.

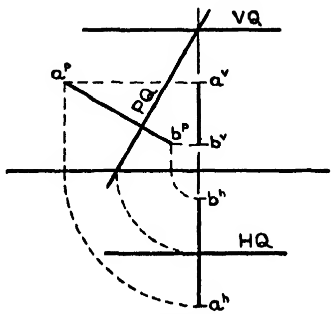

FIG. 157.

113. Perpendicular Planes. If two planes are mutually perpendicular, the fact is not, in general, evident from the traces of the planes. Thus, in Fig. 158, the line A is perpendicular to the plane Q. Hence any plane, as R or S, passed through A, is perpendicular to Q; but this relation cannot be seen directly from the traces of the planes.

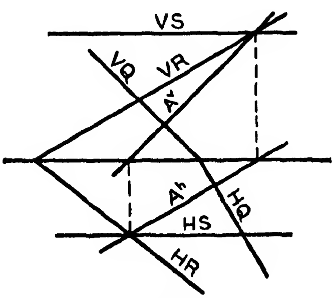

FIG. 158.

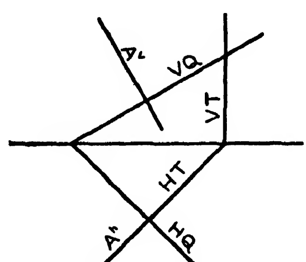

FIG. 159.

But let the line A, Fig. 159, be perpendicular to the plane Q, and let us pass through A the plane T perpendicular to H.

Since HT coincides with A^h, and A^h is perpendicular to HQ, it follows that HT is perpendicular to HQ. Hence we have the proposition:

If two planes are mutually perpendicular, and one of them is perpendicular to a coördinate plane, the traces of the two given planes on that coördinate plane are perpendicular.

114. Lines of Maximum Inclination to *H* and *V*. Let Q be any plane oblique to H. Those of its lines which are perpendicular to HQ have the peculiarity that they make with H a greater (acute) angle than any other lines in the plane. For this reason they are called the **lines of maximum inclination** to H of the plane Q. Let M, Fig. 160, be one of these lines. Since M is perpendicular to HQ, it follows that M^h must also be perpendicular to HQ.

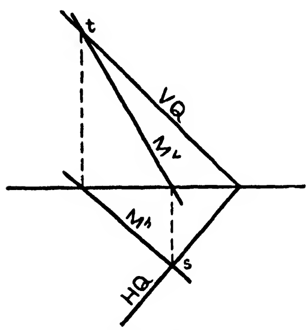

FIG. 160.

A plane is determined when one of its lines M (M^h, M^v) of maximum inclination to H is given. For suppose that s and t, Fig. 160, are the traces of this line. Then the H-trace, HQ, of the required plane passes through s and is perpendicular to M^h, while the V-trace, VQ, is determined by t and the point in which HQ cuts the ground line.

A similar analysis applies to lines of maximum inclination to V.

115. Planes Perpendicular to a Given Line or Plane. One plane, and one only, may be found which contains a given point and is perpendicular to a given straight line. The point may be on the line, or at any distance from it.

One plane, and one only, may be found which contains a given straight line and is perpendicular to a given plane. An exception occurs when the line is itself perpendicular to the given plane, in which case an infinite number of planes may be found.

Problem 10. ***To find the plane which contains a given point and is perpendicular to a given line.***

Analysis. Through the given point draw an auxiliary line which shall be perpendicular to (but does not necessarily intersect) the given line. Through this line pass the required plane perpendicular to the given line.

Construction (Fig. 161). Let c be the given point and A the given line. Through c draw the auxiliary line M, making M^h perpendicular to A^h and M^v parallel to GL. Then the line M is parallel to H, and consequently perpendicular to A (§ 110). Find the V-trace, t, of M. Through t draw VQ perpendicular

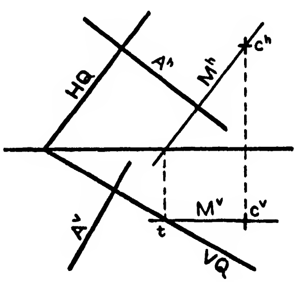

FIG. 161.

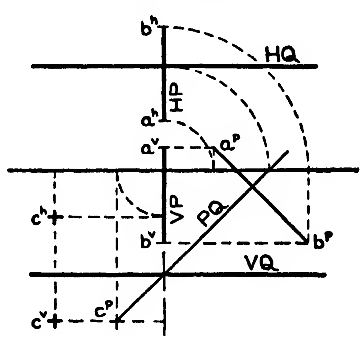

FIG. 162.

to A^v, and from the point in which VQ intersects GL draw HQ perpendicular to A^h. Then Q is perpendicular to A (§ 112) and is the required plane.

Note that the auxiliary line M is a horizontal principal line of the required plane Q. The construction could also be effected by using an auxiliary line parallel to V; such a line would be a vertical principal line of Q.

SPECIAL CASE. The general method fails when the given line is a profile line. Let ab, Fig. 162, be the given line, and c the given point. The required plane Q will be parallel to the ground line, hence perpendicular to P. Find the profile projection of the line ab and the point c. Through c^p draw the *edge view* and profile trace, PQ, of Q, perpendicular to a^pb^p. From PQ find HQ and VQ.

Problem 11. ***To find the plane which contains a given line and is perpendicular to a given plane.***

Analysis. Through any point of the given line draw an auxiliary line perpendicular to the given plane. Any plane containing this line will be perpendicular to the given plane (§ 113). Hence the required plane is the plane determined by the given and auxiliary lines.

Construction (Fig. 163). Let A be the given line and Q the given plane. Assume any point c on the line A, and through

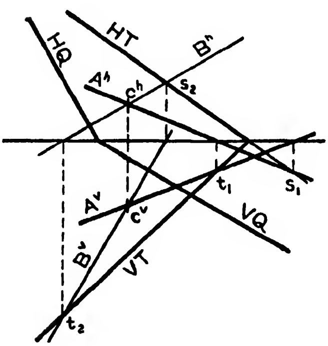

FIG. 163.

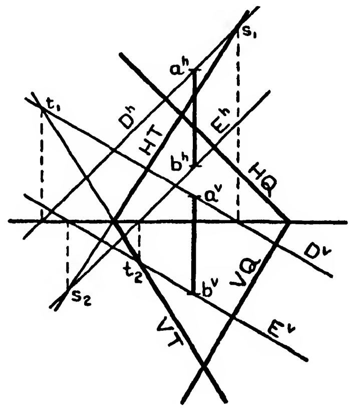

FIG. 164.

c draw the auxiliary line B perpendicular to Q (§ 112). Find the plane T, containing the lines A and B (Prob. 6, § 106); this is the required plane.

SPECIAL CASE I. If the given line is a profile line, the general method applies; but the use of a profile projection may be avoided as follows. Let ab, Fig. 164, be the given line, and Q the given plane. From each end of the given line draw the respective auxiliary lines D and E, each perpendicular to Q. Pass the required plane T through the parallel lines D and E (compare Fig. 145).

SPECIAL CASE II (Fig. 165). Let the given plane, Q, be parallel to the ground line, and let A be the given line. As in

the general case, assume any point c in the line A, and draw the line B perpendicular to Q. Then the required plane, T, is the plane determined by the lines A and B. To find the traces of T we need the traces of these two lines. The traces of A are readily obtained (Prob. 1, § 37). To find the traces

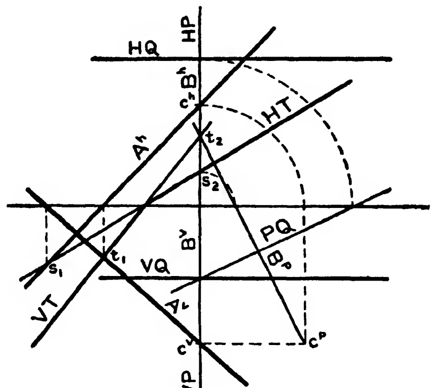

FIG. 165.

of B, we note that B is a profile line and that we know only one point, c, in this line. Nevertheless, B is a definite line in space, since its direction, perpendicular to the plane Q, is known. Hence, find the profile projection, c^p, of the point c, and the profile trace, PQ, of the plane Q. Then the profile projection, B^p, of line B is drawn through c^p perpendicular to PQ. From B^p we can find the H- and V-traces of line B. These traces, together with the traces of line A, determine the traces of the required plane T.

CHAPTER XII

INTERSECTION OF PLANES AND OF LINES AND PLANES — APPLICATIONS

116. Intersecting Planes. Let two planes Q (HQ, VQ) and R (HR, VR) intersect in a line A. The figure will be left for the student to draw. One point of the intersection will be determined if we find where a line in one plane intersects a line in the other. A second point may be similarly found by the intersection of a second pair of lines. Evidently any such pair of lines cannot be chosen at random, for in that case they probably would not intersect. The H-traces of the two planes, however, as they are both in H, will in general intersect. Likewise the two V-traces will in general intersect. But the intersection of the H-traces will be the H-trace of the required line A (§ 96), and the intersection of the V-traces will be the V-trace of A. The projections of the line A may then be determined (Prob. 2, § 37).

117. A Point in the Intersection of Two Planes. The general method of obtaining a point in the line of intersection of two planes is as follows: Let two planes Q and R, as before, intersect in a line A, and let X be any plane not parallel to A. Then X will intersect Q in a line xq, and will intersect R in a line xr. Since these lines are in the same plane, X, and cannot be parallel (why?), they intersect in a point, xqr, which lies in all three planes, X, Q, and R. Hence xqr is a point in the line of intersection, A, of Q and R.

118. The Line of Intersection of Two Planes. The line of intersection of two planes becomes known: (1) when two points of the line are known; (2) when one point and the direction of the line are known. (Compare § 108.)

Problem 12. *To find the line of intersection of two planes.*

Analysis. The traces of the required line of intersection lie at the intersection of the corresponding traces of the given

planes (§ 116). The line may now be determined from its traces (Prob. 2, § 37).

Construction, General Case (Fig. 166). Let Q and R be the given planes. The construction should be evident from the analysis.

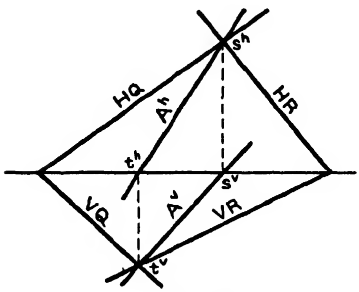

Fig. 166.

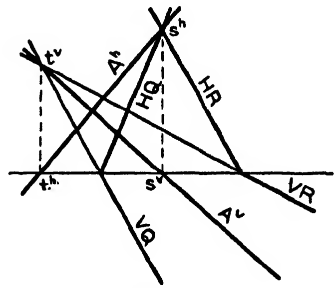

Fig. 167.

A second example, with planes differently situated, is given in Fig. 167.

The general case may fail because of (a) parallel lines; (b) intersections inaccessible; (c) points s and t coincident.

Special Case I. Suppose that one pair of traces is parallel (Fig. 168). Let Q and R be the given planes, with HQ and HR parallel. The intersection of VQ and VR gives the V-trace, t, of the required line of intersection A. Consider the planes Q, R, and H. Since the intersections of H with Q and R are parallel, H must be parallel to the line of intersection, A, of these planes. Hence A^h passes through t^h and parallel to HQ and HR, while A^v passes through t^v and is parallel to GL. The line of intersection is thus seen to be a horizontal principal line (§ 99) of each of the given planes.

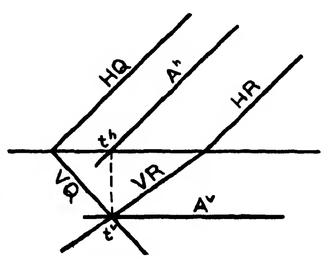

Fig. 168.

If the V-traces of the given planes were parallel, while the H-traces intersected, the resulting line of intersection would be parallel to V, a vertical principal line of each of the given planes.

If both the *H*- and *V*-traces of the given planes are respectively parallel, the planes are parallel (§ 103), except in the case in which each plane is parallel to the ground line.

Special Case II. Suppose that both planes are parallel to the ground line (Fig. 169). Let *Q* and *R* be the given planes.

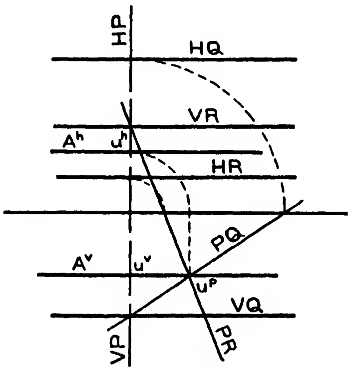

Fig. 169.

One point of the line of intersection may be obtained by finding its profile trace on any profile plane of projection. Let *P* (*HP*, *VP*) be any assumed profile plane. Find the profile traces *PQ* and *PR* of the given planes (§ 60). Their intersection gives u^p, the profile trace of the required line of intersection *A*. From u^p obtain u^h and u^v, one point in the line *A*. A second point is not necessary, since this line must be parallel to both *H* and *V* (Case 1), that is, parallel to the ground line.

Special Case III. One plane is parallel to *H* or *V*, the other is oblique (Fig. 170). Let *Q* and *R* be the given planes,

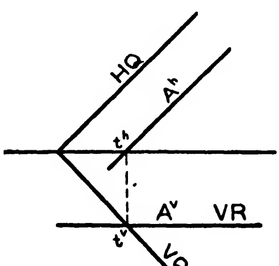

Fig. 170.

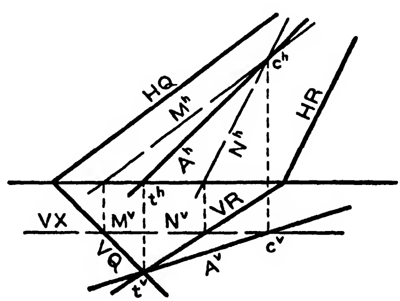

Fig. 171.

the plane *R* being parallel to *H*. The plane *R* therefore intersects *Q* in a horizontal principal line of *Q* (§ 99).

If the plane *R* be taken parallel to *V*, the intersection becomes a vertical principal line of *Q*.

This is an important case, since, on account of the simplicity of the intersection, planes parallel to *H* or *V* are often introduced as auxiliaries when the general case fails.

Special Case IV. The traces are not parallel, but one or both pairs fail to intersect within the limits of the drawing. Let *Q* and *R*, Fig. 171, be the given planes. The intersection of the *V*-traces of these planes gives the point *t*, as in the general case. Since *HQ* and *HR* do not intersect within reach, pass the auxiliary plane, *X*, parallel to *H*. Planes *X* and *Q* intersect in the line *M* (Case III); planes *X* and *R* intersect in the line *N*; the lines *M* and *N* intersect in the point *c*, which is a point in the intersection of the planes *Q* and *R* (§ 117). The points *t* and *c* determine the required line of intersection *A*.

In Fig. 172, neither pair of traces intersect within the limits of the figure. Two auxiliary planes, *X* and *Y*, are used, each of which locates a point on the line of intersection *A*. The auxiliary planes may be parallel to either *H* or *V*; in the figure they are taken parallel to *V*.

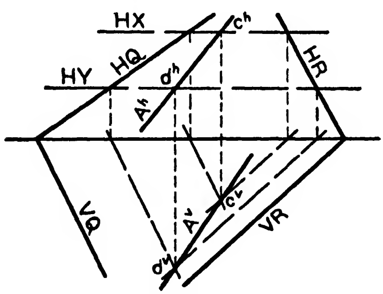

Fig. 172.

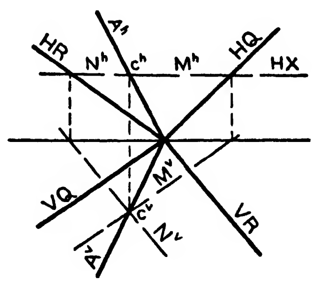

Fig. 173.

Special Case V. Both planes intersect the ground line at the same point (Fig. 173). Let *Q* and *R* be the given planes. The general case fails because the points *s* and *t* are coincident. A point, *c*, in the required line of intersection, *A*, may be determined by means of the auxiliary plane *X* parallel to *V*, as in Fig. 172. An auxiliary plane parallel to *H* may be used if preferred.

SPECIAL CASE VI. One of the planes contains the ground line (Fig. 174). Let Q and R be the given planes. R passes through the ground line, and is determined by the quadrants through which it passes and its angle with H. As in Special Case V, the points s and t are coincident on the ground line. An additional point in the required line of intersection may be found by the use of an auxiliary profile plane of projection, assumed anywhere except through the coincident points s and t. Find the profile traces, PQ and PR, of the given planes (§ 60). These traces intersect in c^p, which is the profile projection of a point in the required line of intersection.

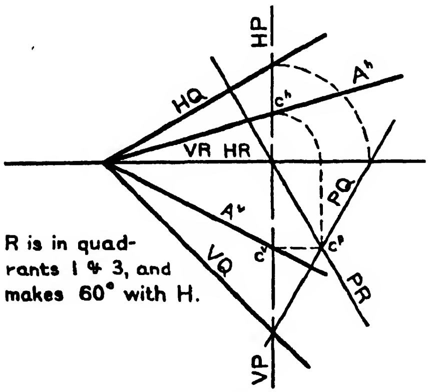

FIG. 174.

119. The Intersection of a Line and a Plane. Let a line A (Fig. 175) intersect a plane Q. The point of intersection will

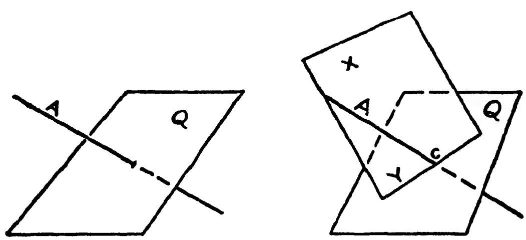

FIG. 175.

be determined if we find where A intersects a line in plane Q. This line cannot, however, be any line chosen at random in Q, for such a line will probably not intersect A. Let a plane, X, be passed through the line A. Then X will intersect Q in a line, Y. This line Y is a line in the plane Q, which is inter-

sected by A at the point c. Hence c is the required point in which A intersects Q. The solution thus depends directly upon the preceding problem.

Problem 13. ***To find the point in which a straight line intersects a plane.***

Analysis. Through the given line, pass any auxiliary plane. Find the line of intersection of the auxiliary plane with the given plane, and note the point in which this line intersects the given line. This is the required point of intersection of the given line and plane.

General Method. Construction (Fig. 176). Let A be the given line, and Q the given plane. Through A pass any auxiliary

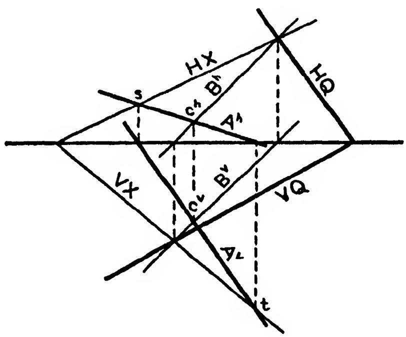

FIG. 176.

plane, as X (§ 98). Find the line of intersection, B, of X and Q (Prob. 12, § 118). The line A intersects the line B in the point c, the required point of intersection of A and Q.

Check. The projection c^h is determined by the intersection of A^h and B^h; c^v is independently determined by the intersection of A^v and B^v. But c^h and c^v are two projections of the same point, and hence must lie in the same projector.

Since the auxiliary plane X may be any plane passed through the line A, ease of construction often depends upon a judicious choice of this plane. Ordinarily, the simplest construction results when the auxiliary plane is taken perpendicular to H or V (see the planes X and Y, Fig. 125).

Usual Method. Construction (Fig. 177). Let A be the given line and Q the given plane. The auxiliary plane X is taken through A and perpendicular to H. Planes X and Q intersect in the line B. The projections A^v and B^v intersect in c^v, one projection of the required point. The projections A^h and B^h are coincident, so that their intersection is indeterminate. Consequently the projection c^h must be located by projecting from c^v.

The auxiliary plane X might otherwise have been taken perpendicular to V, in which case c^h would be determined by the direct intersection of two projections, while c^v would need to

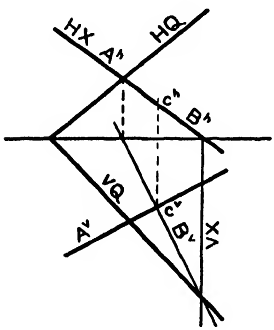

Fig. 177.

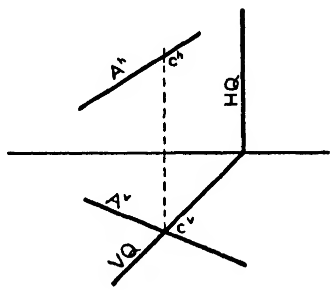

Fig. 178.

be located by projection from c^h. The usual method does not give the check on the construction which appears in the general method. In applications, however, the gain in simplicity usually more than compensates for this.

Special Case. The given plane is perpendicular to H or V. Let the given plane Q, Fig. 178, be perpendicular to V. No construction is necessary, since VQ is an edge view of the plane, and in the V-projection the point in which the line A pierces the plane appears directly. The H-projection of this point is obtained by projecting from the V-projection.

120. The Shortest Distance from a Point to a Plane. The shortest distance from a given point to a given plane may be obtained by dropping a perpendicular from the point to the plane, and then measuring the length of this perpendicular.

Problem 14. ***To find the shortest distance from a point to a plane.***

Analysis. From the given point drop a perpendicular to the given plane. Find the foot of the perpendicular, that is, the point in which the line pierces the given plane. Obtain the true length of the perpendicular.

NOTE. Observe that this solution is a direct application of the previous Problem.

Construction (Fig. 179). Let a be the given point, and Q the given plane. From a draw the indefinite line, C, perpendicular to Q (§ 111). Find the point, b, in which C intersects Q

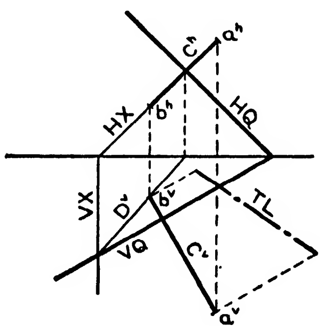

FIG. 179.

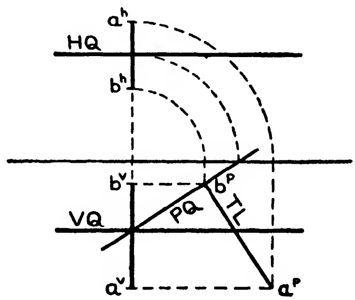

FIG. 180.

(Prob. 13, § 119); in the figure, this is done by using the auxiliary plane X, perpendicular to H. Then a^hb^h and a^vb^v are the projections of the required shortest distance, the true length of which may be found by Problem 3 (§ 78 or § 80).

SPECIAL CASE (Fig. 180). The given plane Q is parallel to the ground line. The required perpendicular from a is evidently a profile line, and may be drawn directly in the profile projection as soon as the profile views of the given point and and plane are obtained. In this case it is not necessary to have the H- and V-projections of the perpendicular in order to know its actual length; these projections are, however, usually considered a part of the problem, and are obtained by projecting from the profile view.

121. The Projection of a Point or Line on a Plane. The projection of a point on a plane is the foot of the perpendicular dropped from the point to the plane. This definition is not confined to the coördinate planes of projection, but applies to any plane in space. However, when a point is projected on to some oblique plane represented by its traces on H and V, the projection must in turn be represented by its projections on H and V. Thus, in Figs. 179 and 180, the point b (b^h, b^v) is the projection of the point a (a^h, a^v) on the plane Q (HQ, VQ).

The projections of a straight line may be obtained by projecting any two of its points, and then connecting these projections.

Problem 15. *To project a line on an oblique plane.*

Analysis. From any two points on the given line, drop perpendiculars to the given plane. Find the points in which

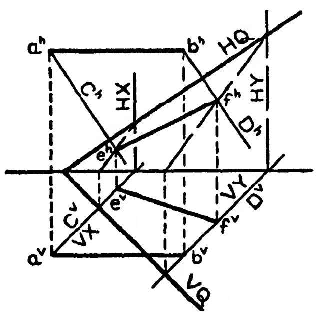

FIG. 181.

these lines intersect, respectively, the given plane, and connect these points.

Construction (Fig. 181). Let it be required to project the line ab on the plane Q. From a draw the line C (C^h,C^v) perpendicular to the plane Q (§ 111), and find the point, e, where C intersects Q (Prob. 13, § 119). From b draw the line D perpendicular to Q, and find point f, where D intersects Q. Then the line ef (e^hf^h, e^vf^v) is the projection of ab on Q.

A second example is given in Fig. 182, the lettering and explanation being the same as for the preceding figure. The

fact that the projection *ef* crosses the given line *ab* shows that this line intersects the plane *Q* at the point *n*, where *ab* and *ef* intersect. Note that the projections n^h and n^v are independ-

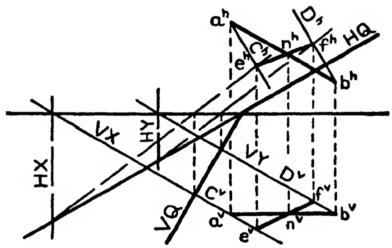

Fig. 182.

ently determined, and should check by lying in the same projector.

Special Case. In Fig. 183 the given line *A* is known to be parallel to the given plane *Q* (§ 104). Hence, to project *A* on

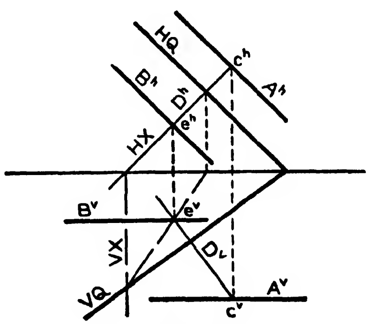

Fig. 183.

Q it is sufficient to project one point of *A*, as for example, point *c*, which projects at point *e*. Then the required line *B* is drawn through *e* parallel to the line *A*.

CHAPTER XIII

INTERSECTIONS OF PLANES AND SOLIDS BOUNDED BY PLANE FACES

122. A Plane Determined by Two Lines. A plane is completely determined when any two of its lines, not necessarily its traces, are known (§ 106). Hence, if a plane be given by means of any two intersecting or parallel lines in space, it is not always necessary, nor even desirable, to find its traces in the solution of a problem in which such a plane occurs.

123. The Intersection of a Line with a Plane Determined by Two Lines. Let it be required to find the point in which the

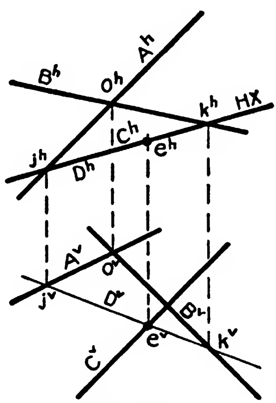

FIG. 184.

line C, Fig. 184, intersects the plane of the intersecting lines A and B. Instead of finding the traces of the plane containing A and B (Prob. 6, § 106), and then finding the point in which C intersects this plane (Prob. 13, § 119), let us proceed at once, as in the usual method of Problem 13, to pass through the line

C an auxiliary plane perpendicular to H (or to V). The line HX, coincident with C^h, is the H-trace and edge view of such a plane. The plane X intersects the line A in point j (see Fig. 178) and the line B in point k. The line D, connecting j and k, must therefore be the line of intersection of the plane X with the plane of the lines A and B. The projection D^v intersects C^v at e^v, which, for the same reasoning as that given in Problem 13, must be one projection of the point in which C intersects the plane of A and B. Finally, e^h is found by projecting from e^v.

Note that in this solution the position of the ground line is not essential. It may therefore be omitted.

Figure 185 shows how to apply this method to find the point in which the profile line cd pierces the plane of the in-

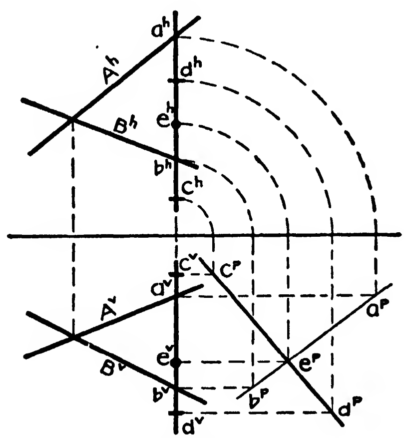

FIG. 185.

tersecting lines A and B. Pass through cd a profile plane. The plane of A and B intersects the profile plane in the line ab. By means of a profile projection it is readily found that the lines ab and cd intersect in the point e, the required piercing point. In this figure the ground line is not omitted, since it is necessary in order to find the profile projection. Nevertheless the positions of the projections e^h and e^v are independent of the position of the ground line.

124. The Intersection of Two Limited Plane Surfaces. Let it be required to find the intersection of the triangle *abc*, Fig. 186, with the plane, indefinite in length, but limited in width by the parallel lines *J* and *K*. The intersection can be found, without finding the traces of either plane, by applying the preceding method, as follows. Using the auxiliary plane *X* which contains *J* and is perpendicular to *H*, we find that the line *J* intersects the plane of the lines *ab* and *bc* in the point *d*. Using the auxiliary plane *Y*, containing *K*, this line intersects the plane of the lines *ac* and *bc* in the point *e*. Therefore the plane *JK* intersects the plane *abc* in the line *de*.

125. Visibility. If both the plane surfaces of Fig. 186 are considered to be opaque, each of them must hide a portion of the

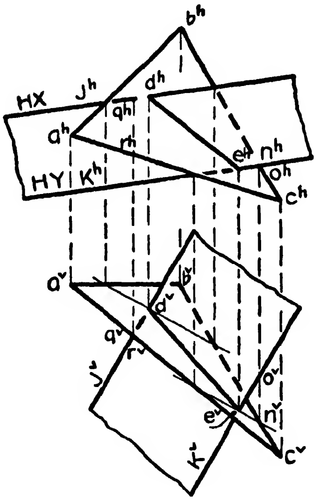

FIG. 186.

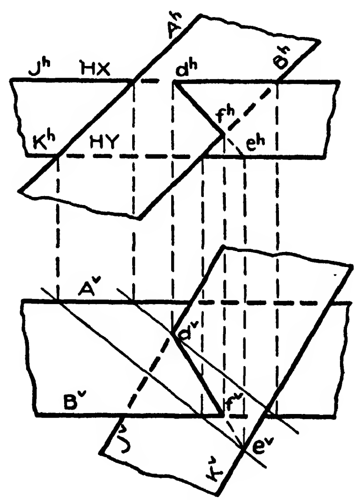

FIG. 187.

other. The visibility of either projection must be determined by means of information obtained from the other projection.

Thus, to determine the visibility of the *H*-projection, take any point in which intersect the projections of two lines not in the same plane. For example, consider the point where K^h intersects b^hc^h. This is actually the projection of two points, *n* in

the line bc, and o in K. Project to the V-projection, obtaining n^v and o^v. From these projections we see that the point o is higher than the point n; that is, K passes above bc. Therefore, in the H-projection, K^h, which contains o^h, is the visible line at the point under consideration.

The other points in the H-projection where the projections of lines not in the same plane cross each other can be tested in the same way. This process finally results in the visibility shown in the figure.

We may reason also as follows: the line K has been found to be above the triangle at the point o. Now K intersects the triangle at the point e. Therefore beyond e the line K passes out of sight under the triangle, so that at the point where K^h intersects a^hc^h, the latter must be the visible line; and so on around.

The visibility of the V-projection is similarly determined. Begin at any point where the two projections of lines not in the same plane cross each other, as for instance where J^v intersects a^vc^v. Project to the H-view to find which line is in *front* of the other. In this case, point r in ac is in front of point q in J, therefore a^vc^v is visible, and J^v is hidden by the triangle. And so on until the complete visibility is found.

126. The Intersection of Two Planes Limited in One Direction. Figure 187 represents the intersection of two planes, each limited in width but indefinite in length. This case presents one new feature over Fig. 186. Using auxiliary planes perpendicular to H, the line J intersects the plane AB in the point d. But the line K does not intersect the plane AB within the limits of its extent.

If, however, the plane AB extended indefinitely beyond B, K would intersect this plane in the point e, outside of B, and de (d^ve^v, d^he^h) would be the line of intersection of the plane JK with such a plane. Therefore, the portion df of the line de, which lies within the limits of the plane AB, must be the intersection required.

The visibility of the two projections is obtained as explained above.

127. The Intersection of a Plane and a Pyramid. Before beginning to find the intersection, visualize the solid to determine its visible faces. Then when the first line of intersection is found, if it be on a visible face, make it a full line. If it is on an invisible face, make it a dotted line; and so on for the complete intersection. The evident advantage of this is that the student will be able to visualize his work more readily and more completely as he goes along. The construction of the

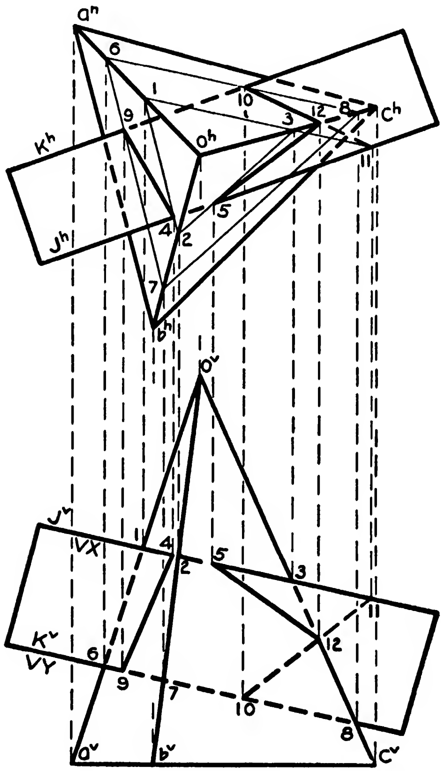

FIG. 188.

intersection requires merely the application and extension of the preceding method.

In Fig. 188 let it be required to find the intersection of the limited plane *JK* with the faces of the triangular pyramid

oabc. In the plan, all the faces are visible except the base *abc*; in the elevation *oab* and *obc* are visible, while *oac* is not visible. In beginning to find the intersection, take the lines *J* and *K* separately. Select any face of the pyramid, as *oab.* Find the point 4 where *J* intersects the face *oab*, by using the plane *X* perpendicular to *V.* Similarly find 9, the point where *K* intersects the same face. The line 4–9 is then the intersection of plane *JK* with face *oab*, and is made a full line in both views, as *oab* is a visible face in both plan and elevation. Taking another face of the pyramid, as *oac*, and proceeding as before, *K* is found to intersect at point 10, while *J* intersects at the point 11, in the plane of *oac*, but outside the face itself. Here it is well to note that while the actual face of the pyramid is of limited extent, the *plane* of the face is unlimited. Joining 10 with 11, the part from 10 to 12 is the actual intersection of *JK* with the face *oac*, is a full line in plan, and dotted in elevation. The line *J* has been found to enter the pyramid at point 4, but the point where it comes out has not yet been found. By inspection of the plan, *J* will necessarily come out on face *obc.* The construction gives 5 as the point where *J* pierces this face. The intersection of *KJ* with the face *obc* will be the line 5–12, for as 12 is on the edge *oc* of the pyramid, it must be a point common to the intersections on the adjoining faces *oca* and *ocb.* The line 5–12 is visible in both plan and elevation.

It should be observed that since point 12 is the intersection of plane *JK* with the edge of the pyramid, it might have been obtained directly by finding where *oc* intersected the plane *JK.* This would have been a convenient construction in case the point 11 had fallen outside the limits of the drawing.

Completing the Visibility. With the intersection itself properly lined in with full and dotted line, the visible portion of each edge of the pyramid or plane can usually be determined by inspection. In case of doubt, however, the relative position of two edges which apparently intersect in one view may be determined by projecting the point of apparent intersection to the other view (§ 125).

128. The Intersection of Two Solids Bounded by Plane Faces. By further extending the preceding methods, the intersections with each other of the surfaces of any two solids bounded by plane faces — a broken line or lines usually known as intersection of the solids — can be found. The detail of the construction varies in each particular case, but involves nothing which has not been already explained. In order not to get lost in the construction, however, it is necessary to proceed in a systematic and orderly manner; therefore, for the benefit of students who may wish to follow through such a construction before attempting one for themselves, we shall give the entire detail of one typical construction.

The two solids chosen, Fig. 189, are a triangular right prism whose long edges are the lines *J*, *K*, and *L*, and a wedge, with a rectangular base *adbe*, and an edge *cf* parallel to the plane of the base. Before starting to draw, we should notice from the *H*-projection that the ends of the prism *JKL* are not concerned in the intersection at all. Therefore it will be well to begin by finding where the lines *J*, *K*, and *L*, considered separately, intersect the other solid. We may notice also, if we wish, that the face *def* of the wedge will not be intersected by the prism. Again, it is evident that the line *ad* will not intersect the prism. This information, however, is not of particular importance at the start.

Before beginning to draw, it is well also to determine the visible faces of each solid considered alone. Then, when any line of the intersection is found, it can be visible only when it is the intersection of two visible faces, one of each solid. The visibility of the intersection being thus determined, and the visible faces of each solid taken alone being known, the visible lines in the combined figure usually can be determined by inspection. In the case in hand, the *V*-projection shows that *ad* is the highest edge of the wedge, hence *ad* is visible in plan. Therefore *df* and *de* are also visible. It follows that the faces *adfc*, *adeb*, and *dfe* are visible in plan. The plan shows *cf* as the extreme back edge, hence in elevation this edge is invisible. This means that *adeb* is the only visible face in this view. For

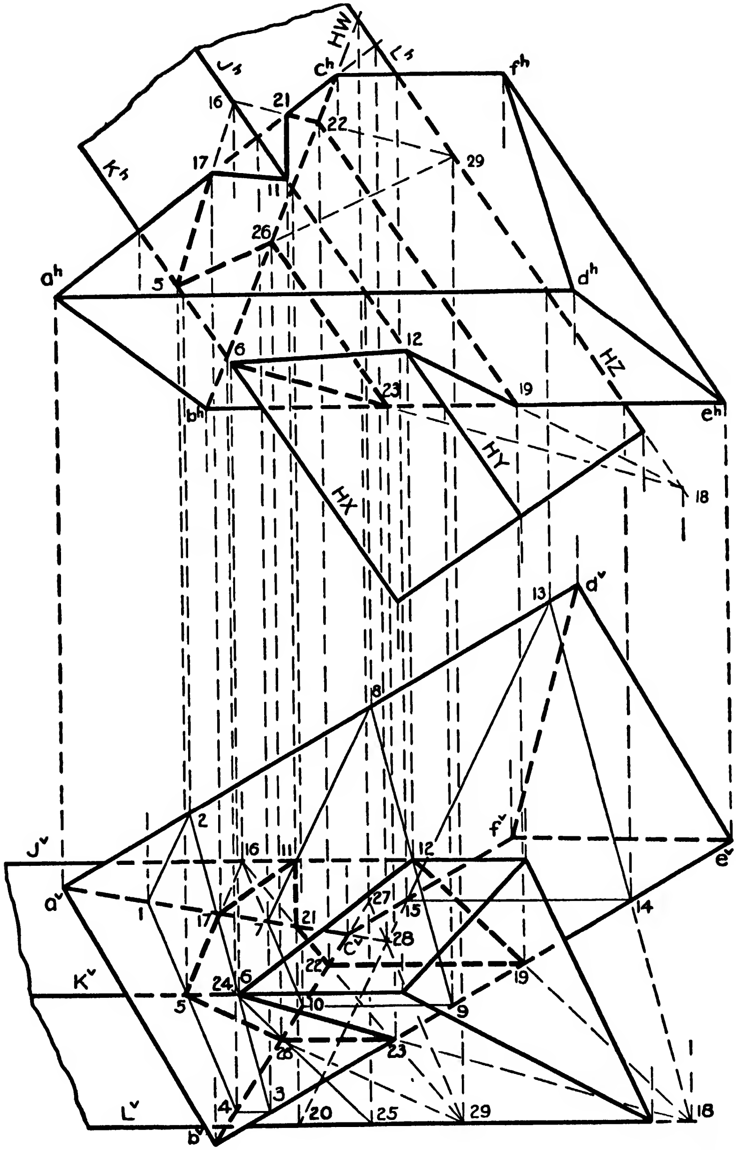

Fig. 189.—The Intersection of Two Solids Bounded by Plane Faces.

the other solid, J, the highest edge, is visible in plan, so the faces JL and JK are seen in plan. In elevation, JK and KL are the visible faces.

To find the actual intersection. Choose any face of the wedge, as *adeb*, and find where all three edges K, J, and L in turn intersect this face. The plane X, perpendicular to H, passed through K, intersects *adeb* in the line 2–3, which crosses K (seen in elevation) in point 6; K intersects *adeb* then in point 6. In the same way J is found to intersect *adeb* in point 12; joining 6 and 12 we have the intersection of face KJ with *adeb*. Since both surfaces are visible in both plan and elevation, the line 6–12 is a full line in each view. The line L is found to intersect *adeb* produced in point 18. Join 12–18 and 6–18, retaining only the portions 12–19 and 6–23, and the intersection of the prism KJL with face *adeb* is complete. The line 12–19 will be invisible in elevation and visible in plan, while 6–23 will be visible in elevation and invisible in plan. The intersection of the two solids must evidently continue from the points 23 and 19 on to the adjoining face *bcfe*, hence choose next this face and find where J, K, and L intersect. Plane X through K intersects this face in the line 4–3, almost if not quite parallel to K. This shows that K, and therefore also J and L are nearly parallel to the face *bcfe*, so their points of intersection are inaccessible. Therefore choose next another face, as *abc*. Line K is found to intersect at point 5, and J at point 16 in the face produced. Joining these points, the part 5–17 is the actual intersection of JK with the face, and is invisible in both views. Line L does not actually intersect the face, but will intersect the face produced. The plane Z through L intersects the edges *ac* and *bc* extended in the points 28 and 27, and this line when produced will cut L in 29. Manifestly, however, it would be a very inaccurate construction to use points so close together as 27 and 28 when the line must be produced much beyond either point, and such a construction should not be used without some check on its accuracy. In this case, such a check is readily available. The planes X, Y, and Z are parallel, and their intersections with the plane *abc*

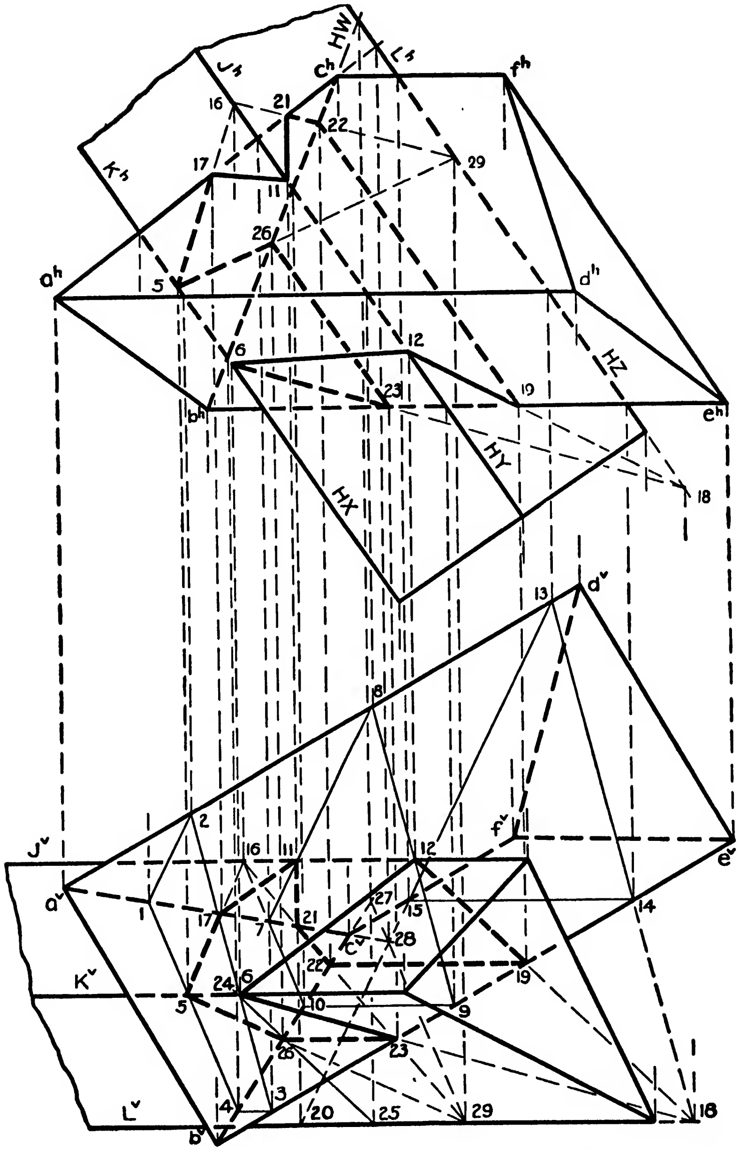

Fig. 189.—The Intersection of Two Solids Bounded by Plane Faces.

must be three parallel lines. Hence, in extending 27–28 to *L* at 29, care should be taken to make the line parallel to the intersections 1–4 and 7–10. Lines *K*, *J*, and *L* intersect the face *abc* or the face produced in points 5, 16, and 29 respectively. Joining these points, the parts 5–17 (already noted), 21–22, and 5–26 form the complete intersection of *KJL* with this face of the wedge. From the points 17 and 21 the intersection must continue on to the adjoining face *acfd*. A little reflection will show that *J* must actually intersect this face. The construction determines the point as 11. As point 17 is in the plane *JK* and also *acfd*, 11–17 joined will be the intersection of *JK* with face *acfd*, and likewise 11–21 is the intersection of *JL* with the same face of the wedge. There remains to be found the intersection on the face *bcfe*. No further points need to be found, however, for as 26 and 23 are both in this face and also in the plane *KL* the line connecting them will be the intersection of *KL* with the face. Similarly 22–19 is the intersection of *JL* with the face, which completes the intersection of the two solids.

Having shown the visibility of the intersection in both views as soon as found, notice that the portions of the edges of the solids which are finally visible may be readily determined by inspection. (The student should take pains to satisfy himself as to the correctness of the visibility of each edge as shown.)

As a check, start with any point of the intersection, as 5; by tracing 5–17–11–21–22–19–12–6–23–26–5, we return to the starting point. In this case there is but one continuous intersection. With different positions of the solids, however, there may be two intersections. This happens, for example, when one solid completely penetrates the other.

129. The Intersection of a Sphere with Another Solid. Suppose that we have two intersecting solids, the first a sphere, the second any solid bounded by plane faces. Pass an auxiliary plane parallel to *H*, so as to cut both solids. This plane will cut from the sphere a circle, and from the other solid a polygon of three or more sides. Both the circle and polygon will project in true shape and size in the *H*-view, where any

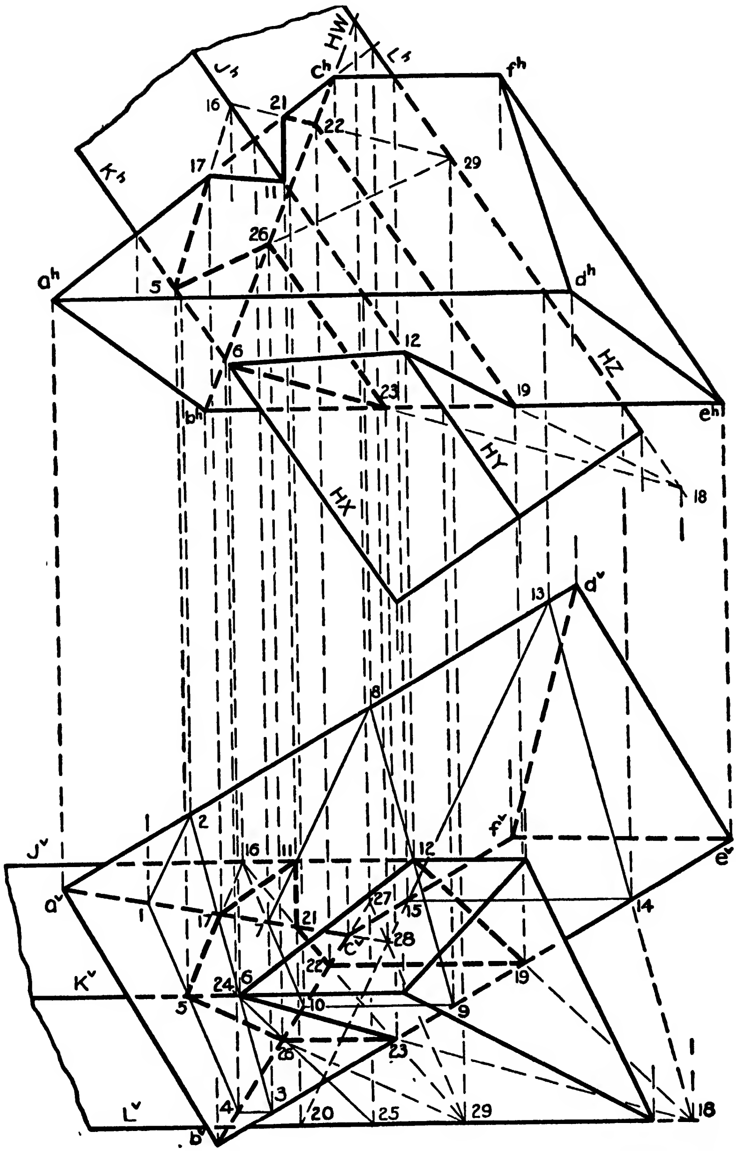

FIG. 189. — THE INTERSECTION OF TWO SOLIDS BOUNDED BY PLANE FACES.

points of intersection may be noted. These points will evidently lie on the line or lines of intersection of the two solids.

An auxiliary plane parallel to *V* may be similarly used, the sections cut from the two solids appearing in true shape and size in the *V*-projection. In any particular problem, therefore, the solution would be effected by planes parallel to *H*, or to *V*, according to convenience. Indeed, it is often desirable to use auxiliary planes in each of these directions.

The above method of solution can be applied to special cases of the intersection of a sphere with some of the curved solids. For example, let the curved solid be a circular right cylinder with its axis perpendicular to *H*. Planes parallel to *H* will cut circles, while planes parallel to *V* will cut rectangles from this cylinder. Hence the analysis previously given can be applied to this case.

In general, if two intersecting solids are such, and so placed that planes parallel to *H* or to *V* will intersect both solids in simple, readily determined figures, such as straight lines or circles, the intersection of the solids may be found by the use of such planes as auxiliaries.

A few simple examples are given in the following Articles.

130. The Intersection of a Sphere with a Prism (Fig. 190). Given the sphere and the triangular right prism *abc–def*, both placed in the first quadrant. It is evident that the intersection is formed by the three vertical sides of the prism, therefore the plan of the intersection coincides with the plan of the prism. The intersection of each vertical side of the prism is here found as explained in § 84. (See Figs. 108 and 109.)

Let us consider, for example, the face *cbfd*. The points 1 and 5 are located at once in the elevation on B^v, the other projection of circle *B*. The plane *R* cuts from the sphere the circle *R*. How is its diameter found? The vertical side of the prism intersects this circle in the points 2 and 8.

The other points, 3, 4, 6, 7, are found in a similar manner, and the ellipse drawn. It may be noted that the ellipse in elevation has its major axis vertical and equal in length to 1^h–5^h, while its minor axis is equal to 1^v–5^v.

The intersections of the other two sides of the prism are found in a similar manner, as in Fig. 190. The *V*-projection of the first intersection is a complete ellipse, but the other two are partial ones, with the points 10 and 11 in common.

Visibility. In any case of the intersection of two surfaces, not only the visible part of the intersection should be shown, but also the visible edges and the outlines of the surfaces, as in § 129.

Visible Intersections. The face first considered, *cbdf*, is the back of the prism, hence its intersection with the sphere is wholly invisible in elevation. The other two faces are each visible in elevation. In each case as much of the intersection as lies on the front or visible half of the sphere will be visible in elevation, that is, 10 to 13 and 11 to 14 on the right-hand curve, 10 to 16 and 11 to 17 on the left-hand curve.

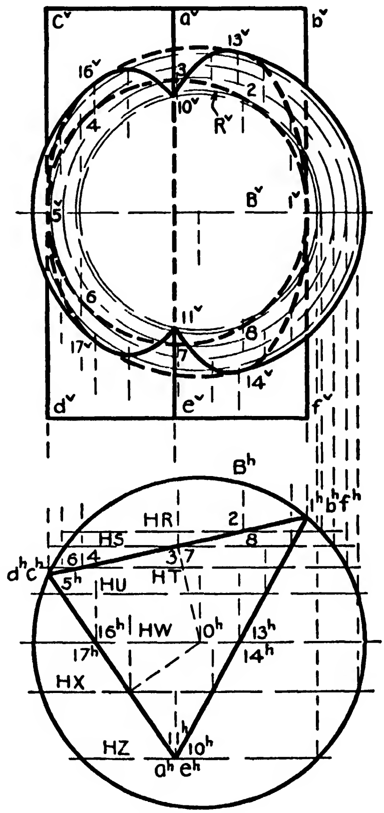

Fig. 190.

Visible Edges and Outlines. The edge *ae* of the prism is the front edge, and intersects the sphere at the points 10 and 11 on the front half. Hence *ae* is visible from a^v to 10^v, and from 11^v to e^v. Since the edge *cd* is tangent to the sphere at point 5 on the back half, *cd* disappears on *V* where it crosses the outline of the sphere. Hence the outline of the sphere is visible from the left toward the right as far as points 16^v and 17^v. Likewise, the edge *bf* passes behind the sphere, and the outline of the sphere is visible from 13^v around to 14^v.

131. The Intersection of a Sphere with a Right Cylinder.

Example 1 (Fig. 191). *Given a sphere and a circular cylinder.* The plan of the intersection is evidently the arc of the circle $1^h5^h7^h$, and a study of this plan view will show that the intersection consists of one continuous curve. The points 1 and 7, shown in plan on the outline of the sphere, are projected directly to 1^v and 7^v. The points 4, 10, 6, 8, must lie on the

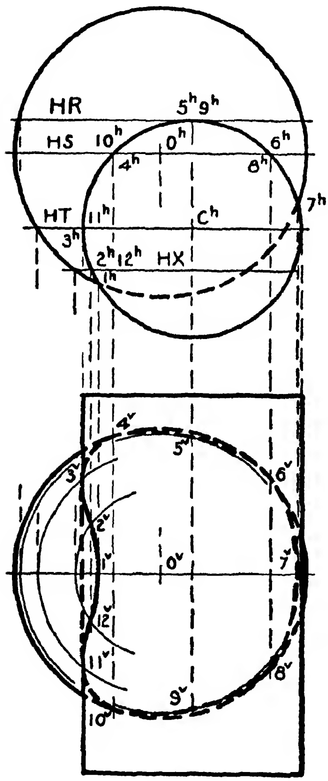

Fig. 191.

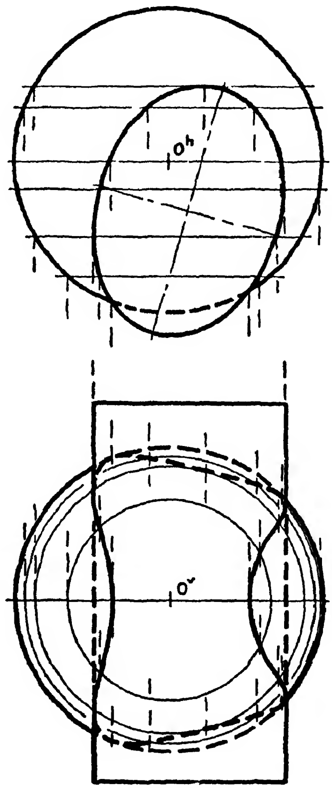

Fig. 192.

great circle of the sphere in elevation, and are projected at once to 4^v, 10^v, 6^v and 8^v. Other points are found by means of auxiliary planes parallel to *V*, as in the preceding figure. The plane *T*, passed through c^h, gives the points 3^v and 11^v, in which

the intersection is tangent to the outline of the cylinder. The visibility of the curve and of the outlines of the surfaces is determined as previously explained.

Example 2 (Fig. 192). *Given a sphere and an elliptical cylinder.* The intersection and the visibility are found as in Example 1. Notice that here two parts of the curve are visible.

Example 3 (Fig. 193). *Given a sphere and a circular cylinder.* In this case an inspection of the plan serves to show that the intersection will consist of two curves, each entirely distinct from the other. Each curve in elevation will be symmetrical with respect to the horizontal center line of the sphere (why?). The solution is left to the student.

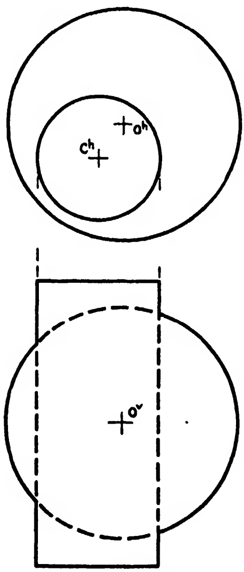

Fig. 193.

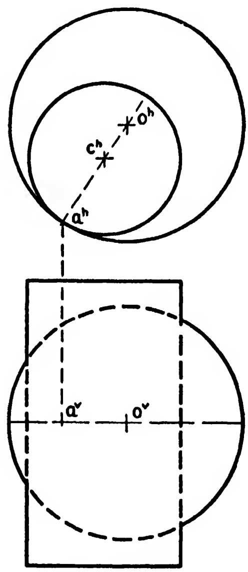

Fig. 194.

Example 4 (Fig. 194). *Given a sphere and a circular cylinder.* In this example it may not be so easy to vizualize the intersection by considering the plan view. Nevertheless a little study will show that there must be two closed figures having a common point *a*. As a matter of fact the intersection when found in elevation will be shaped like an irregular figure

8, the curve crossing itself (not tangent), at a. It is called a one-curve intersection. The solution is left to the student.

132. The Intersection of a Cylinder and a Torus. Another example is shown in Fig. 195. An auxiliary plane, as Q, parallel to H, cuts the cylinder in the elements A and B, and the torus in the circles C and D, giving six points, 1, 2, . . . 6, of the intersection. The chosen secant planes, parallel to H, should

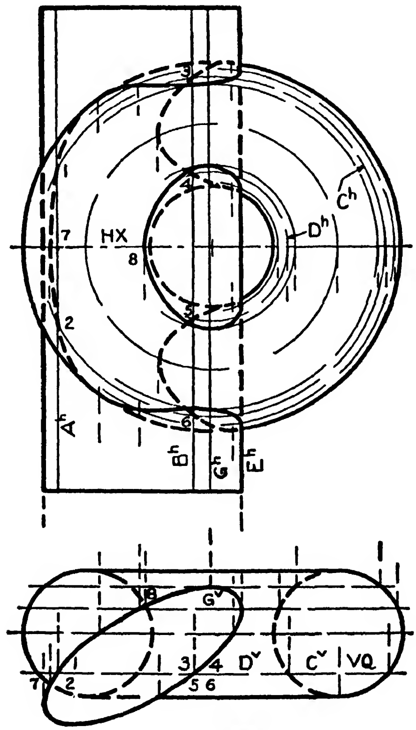

FIG. 195.

include one through the contour element E of the cylinder; one through the highest element, G; one through the center of the torus; and one tangent at the bottom of the torus. The points 7 and 8 of the intersection evidently lie in the plane X, which passes through the center of the torus parallel to V.

CHAPTER XIV

PROBLEMS INVOLVING THE REVOLUTION OF PLANES

133. A Line Lying in a Given Plane. One method of locating a line so that it will lie in a given plane is to assume the traces of the line in the corresponding traces of the plane. This has already been explained in § 97. (See Fig. 123.)

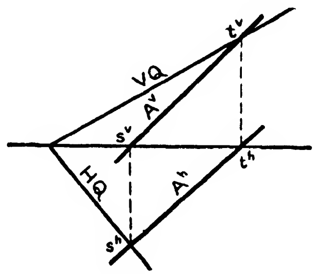

FIG. 123 (repeated).

Another method is to assume one of the projections of the line. As soon as either the *H*- or the *V*-projection of a line is assumed, the line becomes definite from the condition that it shall lie in the given plane.

Problem 16. ***Given one projection of a line lying in a plane, to find the other projection.***

Analysis. The other projection of the line will be determined if two points on it are found. Whenever it is practicable, the simplest points to consider are the two traces, *s* and *t*, which lie, respectively, in the horizontal and vertical traces of the given planes (§ 96). One trace of the line will be at the intersection of the given projection with the corresponding trace of the plane, the other projection of this point being in *GL*. The other trace of the line will be in the other trace of the plane, and is determined by a projector drawn from the intersection of the given projection with *GL*.

Construction (Fig. 196). Let the plane Q and the projection A^h be given. The intersection of A^h and HQ is the H-trace of the line A (§ 96). The V-projection, s^v, of this point lies in GL. The intersection of A^h and GL is the H-projection of the

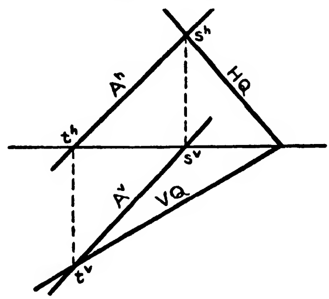

Fig. 196.

V-trace of the line (Prob. 1, § 37). The actual trace, t^v, lies in VQ. The projection A^v is now determined, since two points, s^v and t^v, are known.

Similarly, the projection A^h may be found if A^v is given.

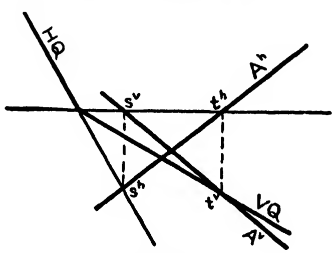

Fig. 197.

A second example is given in Fig. 197. The lettering and explanation are the same as for Fig. 196.

Special Case I. Suppose that the given projection is parallel (a) to the corresponding trace of the plane; or (b) to the ground line. In either event the line should be recognized as one of the principal lines of the plane (§ 99), parallel to H or V as the case may be. Thus, in Fig. 198, if A^h is given parallel to HQ, A^v is parallel to GL; or if A^v is given parallel to GL, then A^h is parallel to HQ. The line A in this case is a horizontal

principal line of Q, having its V-trace at the point t (t^h, t^v). Similarly, if A^v is given parallel to VQ, A^h is parallel to GL, and *vice versa*, the line being a vertical principal line of Q.

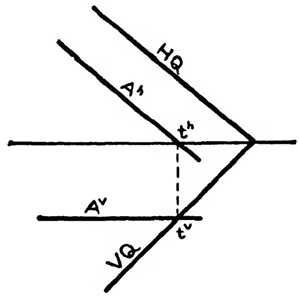

FIG. 198.

SPECIAL CASE II. Suppose that both the plane and the given projection of the line are parallel to the ground line. Whichever projection of the line is given, the other projection will also be parallel to the ground line. Let Q, Fig. 199, be the given plane, and suppose A^h to be given. One point will be

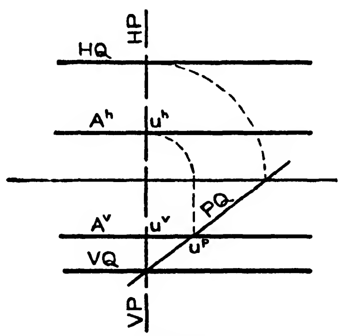

FIG. 199.

sufficient to locate A^v. This point may suitably be the profile trace (§ 59) of the line, on any assumed profile plane, as P. Find first the profile trace, PQ, of the given plane (§ 60). Then from u^h we can project to u^p on PQ, thence to u^v on VP. The projection A^v passes through u^v. The reverse of this construction will locate A^h if A^v is given.

Special Case III. Suppose that the given plane is perpendicular to H or V. Let the given plane Q, Fig. 200, be perpendicular to H. Then if A^v is given, in any position except perpendicular to GL, A^h coincides with HQ, since this trace is an edge view of the plane Q (§§ 51 and 98). In the exceptional position, shown by B^v perpendicular to GL, the H-projection, B^h, reduces to a point on HQ.

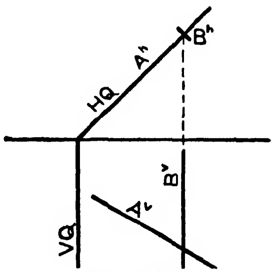

Fig. 200.

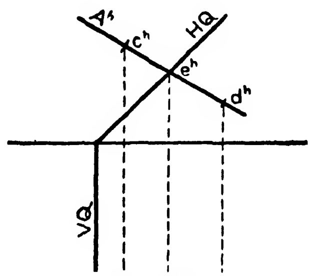

Fig. 201.

Conversely, if A^h is given coincident with HQ, A^v may be taken at random in any position except perpendicular to GL, and the case is indeterminate.

But if the given plane Q, Fig. 201, be perpendicular to H, and if A^h be assumed not coincident with HQ, the solution becomes impossible, for there are no points in the plane Q to correspond with the assumed projections c^h, d^h, or in fact with any point of A^h except the intersection, e^h, of A^h and HQ. The results are similar if the given plane is perpendicular to V.

Corollary I. *Given one projection of a point lying in a plane, to find the other projection.*

Analysis. Through the given point pass any line lying in the plane. Find the other projection of the line. This projection must contain the required projection of the given point.

Construction (Fig. 202). Let Q be the given plane, and assume a^h to be given; it is required to find a^v. Through a^h draw the H-projection, M^h, of some line in the plane Q. In general, the simplest construction results when the assumed line is a principal line of the plane; hence M^h is here drawn parallel to HQ. Find M^v (Special Case I). Then a^v lies on

M^v and is obtained by projecting from a^h. The reverse of this construction will locate a^h when a^v is given.

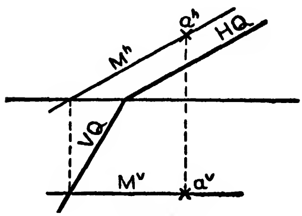

FIG. 202.

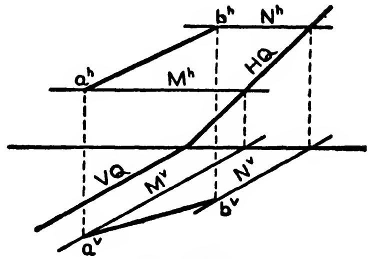

FIG. 203.

COROLLARY II. *To find the second projection of a line lying in a plane when the general solution fails, partially or wholly.*

Let Q, Fig. 203, be the given plane, and let $a^h\ b^h$ be given so that it cannot be produced to intersect either HQ or GL within the limits of the figure. Then a^v and b^v, or in general any two points on the line, may be located by Corollary I.

COROLLARY III. *To find a line of maximum inclination of a plane.*

Let Q, Fig. 204, be the given plane. A line of maximum inclination to H is perpendicular to HQ (§ 114). Hence assume A^h perpendicular to HQ, and find A^v by the general method.

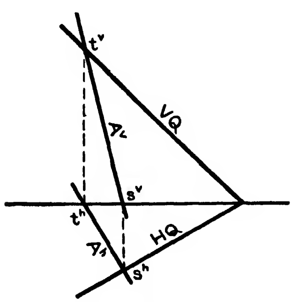

FIG. 204.

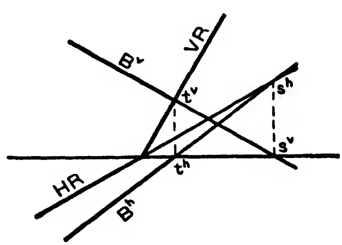

FIG. 205.

In Fig. 205, the line B is in the plane R, and is a line of maximum inclination to V. The projection B^v is taken perpendicular to VR, and then B^h is found.

134. Revolution of a Plane About an Axis Perpendicular to *H* or *V*. As an illustration of the general case of this problem, let it be required to revolve the plane *Q*, Fig. 206, through an angle, α, about the line *A*, perpendicular to *H*, as an axis. The axis pierces *Q* at the point *c* (c^h, c^v). This point is found most readily by observing that since A^h is a point, c^h coincides with A^h. Then c^h is one projection of a point lying in plane *Q*, hence c^v is the corresponding projection (Prob. 16, Cor. 1, § 133). In any revolution of *Q* about *A*, the point *c* will remain fixed. Now a plane is determined when a point and a line are known; hence, having the fixed point *c*, it will be necessary to revolve but one line of *Q*. This line may conveniently be the *H*-trace, *HQ*, of *Q*, which is now revolved about c^h as a center to the required position *HR*. The *V*-trace, *VR*, is then determined by the fact that the plane *R* must contain the point *c*. Consequently *R* contains the line *N*, drawn through *c* and parallel to *HR*. (See Example 3, § 108.)

Let now the plane *Q*, Fig. 206, be further revolved, until the *H*-trace takes position *HS*, perpendicular to the ground line.

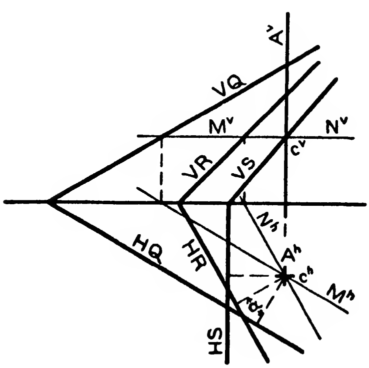

Fig. 206.

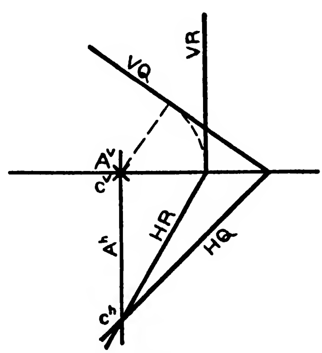

Fig. 207.

In this new position, the plane will be perpendicular to *V*. Hence the *V*-trace, *VS*, will be an edge view of the plane, and must therefore pass directly through c^v.

A plane may be revolved until perpendicular to *H* by using

an axis perpendicular to V. This is done in Fig. 207, where the plane Q is revolved about the line A, perpendicular to V, until the V-trace of the plane, in its new position VR, is perpendicular to the ground line. Then HR is an edge view of the plane, and passes through c^h. In this figure, the axis A, besides being perpendicular to V, is taken lying in H, which simplifies the finding of the point c, in which A intersects Q.

As in the revolution of a straight line (§§ 78, 79), a plane, when revolved about an axis perpendicular to H, maintains always its original angle with H. If the axis is perpendicular to V, the angle between the revolving plane and V does not change.

135. The Distance between Two Parallel Planes. As a simple application of the revolution of planes about axes perpendicular to H or V, we will consider the following Problem.

Problem 17. ***To find the perpendicular distance between two parallel planes.***

Analysis. Assume an axis perpendicular to H (or V), and revolve both planes about this axis until they are perpendicular to V (or H). The perpendicular distance between the planes will then appear.

Construction (Fig. 208). Let Q and R be the given planes. Assume the axis A perpendicular to H, and for convenience lying in V. Revolve both planes about the axis A until they are perpendicular to V (§ 134). Then the new V-traces, VQ_1 and VR_1, are the edge views of the planes, and the perpendicular distance, p, between these traces is equal to the distance between the planes.

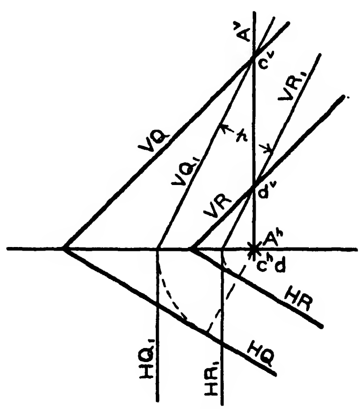

FIG. 208.

Corollary. *To find the perpendicular distance from a point to a plane.*

Analysis. Take an axis perpendicular to H (or V) through the given point. About this axis, revolve the plane until it is perpendicular to V (or H). Then, since the given point on the axis does not move, or change its relation to the plane, the perpendicular distance appears directly in the V (or H) projection.

The construction is left to the student. The problem is the same as Problem 14 (§ 120), a different method of solution having now been found.

136. The Angles between an Oblique Plane and the Coördinate Planes. The angle which a given plane Q, oblique to both H and V, makes with either coördinate plane, for example H, may be found:

(*a*) by drawing in the plane a line of maximum inclination to H (Prob. 16, Cor. 3, § 133) and then finding the angle which this line makes with H (§ 79 or § 82);

(*b*) by finding the edge view of the plane on a secondary plane of projection taken perpendicular to HQ (§ 70);

(*c*) by revolving the plane about an axis perpendicular to H until the plane is perpendicular to V (§ 134).

The various constructions resulting from these different methods resemble each other to a considerable extent, and are essentially the same, so that it is needless to take them all up in detail. The method chosen in this text is thought to be the one which best shows the connection between this problem and the two that follow.

Problem 18. ***To find the angles which an oblique plane makes with H and V.***

Analysis. To find the angle between the given plane and H, assume an axis of revolution lying in V and perpendicular to H. Revolve the plane until perpendicular to V, when the angle between the new V-trace and the ground line will be the angle the plane makes with H (§§ 134, 136). Similarly for the angle between the given plane and V.

Construction (Fig. 209). Let Q be the given plane. Assume the axis A, lying in V and perpendicular to H. Revolve Q about A to the position R (HR, VR), perpendicular to V (§ 134). Then the angle between VR and GL is the angle α, which Q makes with H. Similarly, by revolving about the axis B, lying in H and perpendicular to V, we obtain the angle β, which Q makes with V.

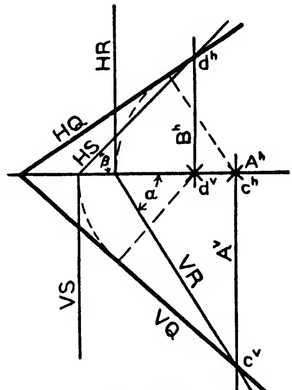

Fig. 209.

It will be seen in Fig. 209 that the traces HR and VS, perpendicular to GL, are not essential to the required angles α and β, and that these traces may therefore be omitted. This is done in Fig. 210, in which are found the angles α and β, which the given plane Q makes with H and V, respectively.

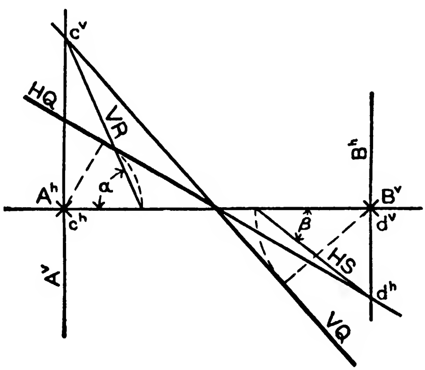

Fig. 210.

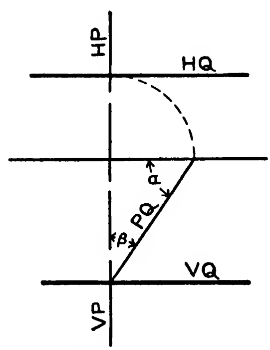

Fig. 211.

Special Case. Let the given plane Q, Fig. 211, be parallel to the ground line. Find the profile trace of Q on any assumed profile plane. This trace, PQ, is an edge view of the plane (§ 60), and gives the required angles, α with H and β with V, directly.

137. Planes making Given Angles with *H* and *V*. The preceding problem gives rise to two converse problems, in which it is required to find planes which make given angles with the coördinate planes.

Problem 19. ***Given one trace of a plane, and the angle which the plane makes with either H or V, to find the other trace of the plane.*** There are two cases, which are stated below.

Analysis. The required result may be obtained by reversing the construction of Problem 18.

CASE I. *Given one trace and the angle which the plane makes with the same coördinate plane.*

Construction (Fig. 212). Let HQ and the angle α, which Q makes with H, be given. Assume the axis A, lying in V and perpendicular to H. Revolve the given trace HQ about this axis to the position HR, perpendicular to GL. Complete the plane in this position by drawing VR at the given angle, α,

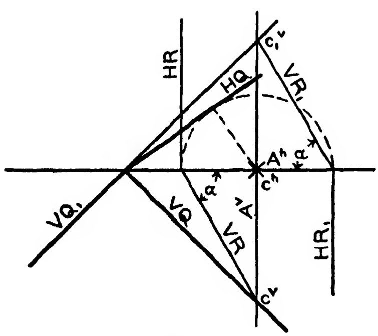

FIG. 212.

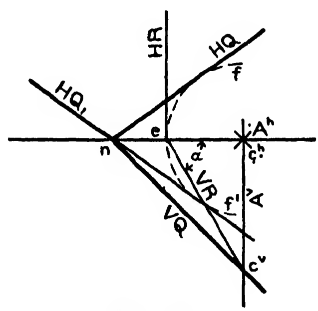

FIG. 213.

with H. Then R intersects the axis A at the point $c\,(c^v, c^h)$. Let R now be supposed to be revolved back about the axis A until HR coincides with HQ; the point c remains fixed. Since c^v is in V, VQ must pass through c^v, and is determined by this point and the point in which HQ intersects GL.

The given trace HQ may also be revolved to the position HR_1, giving a second point c_1^v, and a second result, VQ_1. The distances of c^v and c_1^v from GL are obviously the same, hence VQ and VQ_1 make equal angles with GL.

It is evident in Fig. 212 that the construction is possible for any given angle α between 0° and 90°. If $\alpha = 90°$, but one result is possible, and no construction is necessary. Why?

CASE II. *Given one trace and the angle which the plane makes with the* **other** *coördinate plane.*

Construction (Fig. 213). Let VQ and the angle α, which the plane makes with H, be given. Assume an axis, A, lying in V and perpendicular to H. Since A^v and VQ are both lines in V, they intersect, locating the point $c\,(c^v,\ c^h)$, in which A intersects the plane Q. Place the plane R, perpendicular to V, so as to pass through c, and making the given angle, α, with H. Revolve R about A. In this revolution, the angle α with H does not change (§ 134); hence HQ must be tangent to the arc ef, drawn with c^h as center and radius c^he. This, together with the fact that HQ must pass through the point n, in which VQ intersects GL, determines HQ.

The point e may also be revolved in the opposite direction, along the arc ef', thus giving a second possible result, HQ_1, drawn through n and tangent to ef'. It is evident that HQ and HQ_1 make equal angles with the ground line.

This construction is possible only when the given acute angle α, which Q makes with H, is greater than the acute angle which VQ makes with GL. If α is given as exactly equal to the angle between VQ and GL, no construction is necessary, and only one result is possible. Why?

SPECIAL CASE. The two cases of this problem merge into one when the given trace is parallel to the ground line, the solution being effected by means of an assumed profile plane. (Compare Fig. 211.) There are always two possible results for any given value of α or β greater than 0° and less than 90°.

Problem 20. ***Given the angles which a plane makes with H and V, to find the traces of the plane.***

As in the preceding problems, we shall call the angle with H, α, and the angle with V, β. If any plane can be found which makes the angles α with H and β with V, there is an infinite number of parallel planes making the same angles. The

location of the plane becomes definite only when some point lying in the plane is known. But a plane may be readily transferred, parallel to itself, from one position to another (see Prob. 9, § 107). Hence, the problem as stated will be considered solved when any one of the series of parallel planes is found.

Analysis. It is evident that the construction in Fig. 209, in which α and β are found for a given plane Q, cannot be readily reversed. But consider Fig. 214, in which α and β for the given plane Q are found by taking the axes of revolution, A and B, through the same point, o, in the ground line. Since VR is an edge view of Q, the perpendicular distance from o to Q is equal to the perpendicular ok, dropped from o to VR. Moreover, since HS is an edge view of Q, the perpendicular distance from o to Q is equal to the perpendicular om, dropped from o to HS. Hence $ok = om$, whence it follows that the two edge views, VR and HS, are each tangent to a circle drawn with o as center and radius $ok = om$. The plane Q may now be obtained from the angles α and β by reversing this construction.

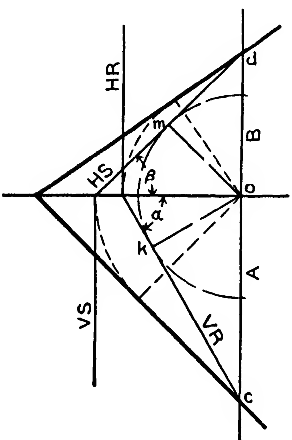

FIG. 214.

Construction (Fig. 215). Assume any point o on the ground line. Through o draw an axis perpendicular to the ground line. With o as center, draw a circle with any assumed radius, the radius being the distance from o to the required plane Q.

Tangent to the circle, draw the edge view VR, which represents the plane perpendicular to V, at the given angle α with H. Now VR intersects the axis at c, and the ground line at e. Hence c is the fixed point on VQ, while HQ must be tangent to the arc ef, drawn about o as center with the radius oe.

Again, tangent to the circle, draw the edge view HS, which represents the plane perpendicular to H, at the given angle β

with V. We find that HS intersects the axis at d, and the ground line at g. Hence, d is the fixed point on HQ, while VQ must be tangent to the arc gj, drawn with center o and radius og.

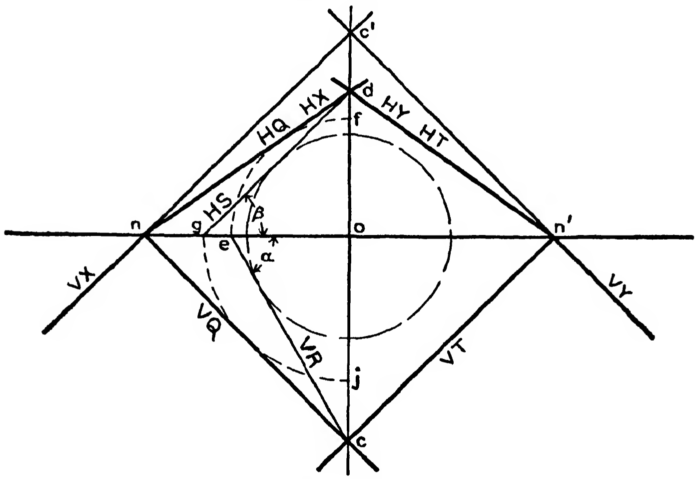

FIG. 215.

We now have one point each on HQ and VQ, and an arc to which each is tangent. Therefore HQ and VQ are determined.

Check. Since HQ and VQ are independently determined, they must intersect the ground line at the same point, n.

If α and β are given as acute angles, as is usual, a solution is possible only when the sum of α and β is not less than 90° nor more than 180°. But when the sum of these angles is greater than 90°, and each angle is less than 90°, four possible non-parallel results may be found.

Thus, having found the plane Q, Fig. 215, making α with H and β with V, make $on' = on$. Then it is evident by symmetry that the plane T, drawn with VT through c and n', and HT through d and n', makes the same angles. Also, make $oc' = oc$, and draw VX through n and c'. Then VX makes the same angle with GL as does VQ, hence taking HX coincident with HQ, plane X makes with H and V the same angles as does Q (compare Fig. 212). A fourth result, plane Y (VY, HY) is obtained by symmetry from plane X.

There are other methods of obtaining the three additional results after the first plane Q is obtained. Thus, a distance $od' = od$ might be laid off below the ground line, and H-traces drawn from d' to n and n'. But after four non-parallel results are obtained, all further results will be parallel to some plane already found. This is because there are but four possible slopes of an oblique plane: (1) downward, forward, to the right; (2) downward, backward, to the right; (3) downward, forward, to the left; (4) downward, backward, to the left. (See § 49.) The slope of a plane is the same as the slope of its lines of maximum inclination to H. These lines, if desired, may be found by Problem 16, Cor. 3, § 133.

Special Case. The sum of α and β is 90°. The plane is then parallel to the ground line, and its profile trace is its edge view. Hence (Fig. 216), assume any profile plane of projection. Draw two non-parallel profile traces making the given angles with H and V, and find the corresponding H- and V-traces. Only two results are possible, corresponding to the two possible slopes: downward and forward, and downward and backward (or upward and backward, upward and forward).

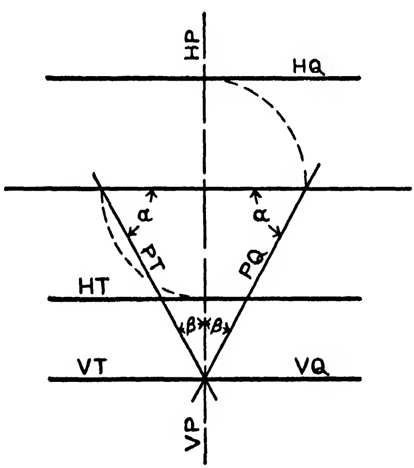

Fig. 216.

Other special cases, which may be solved by inspection, result when (1) either α or β is 0°, in which case the other angle must be 90° (one result); (2) either α or β is 90°, the other angle being greater than 0° and less than 90° (two results); (3) $\alpha = \beta = 90°$ (one result). These cases are left to the student.

CHAPTER XV

OTHER PROBLEMS INVOLVING THE REVOLUTION OF PLANES

138. The Revolution of a Plane about One of Its Traces. A very useful revolution of a plane is that in which the plane is revolved about one of its own traces as an axis. For example, the plane Q may be revolved about HQ as an axis until Q coincides with H. The figure is left for the student to draw. Any points, lines, or figures contained in Q will be carried by this revolution into the coördinate plane, where they will appear in their true sizes and relations.

Problem 21. ***To find the position of a point lying in a plane, when the plane is revolved into H or V about the corresponding trace.***

NOTE. Since only one projection of a point lying in a plane can, in general, be assumed, it may be necessary to find the second projection by the use of Problem 16 (§ 133), before attempting to solve the present problem.

Analysis. Suppose point a lying in plane Q, to be revolved about HQ until it lies in H. The figure is left to the student. The process is the same as described in § 76, and shown pictorially in Fig. 98. The statement of § 76 may be modified to meet the present situation, as follows. If a point a lying in plane Q, be revolved about HQ as an axis, the circular path of the revolving point will project on H as a straight line perpendicular to HQ. If the point a be revolved into H, its revolved position will be at a distance from HQ equal to the true distance of the point from HQ.

An analogous statement applies to the revolution of a point about VQ into V.

Construction (Fig. 217). Let a, lying in the plane Q, be the given point, and let it be required to revolve a about HQ into H. From a^h draw the line a^he^h perpendicular to HQ, and produce the line indefinitely. This line is the H-projection of the circular path of the revolving point, the center of revolution being the point e (e^h) lying on HQ. The line a^he^h is one projection of a line lying in Q (a line of maximum inclination to H, § 114). Find a^ve^v (Prob. 16, § 133), and the true length, a^ve_1, of the line ae (Prob. 3, § 78). The distance a^ve_1 is the actual distance of the point a from the center of revolution e^h. Lay off $e^ha_r = a^ve_1$, thus giving the required revolved position a_r.

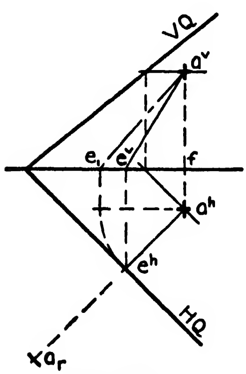

FIG. 217.

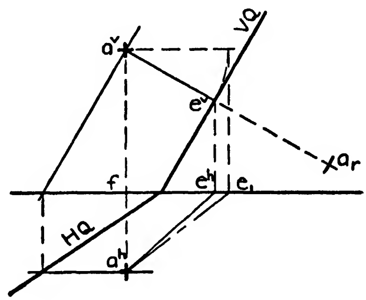

FIG. 218.

NOTE. The revolved position a_r represents the revolution of Q about HQ through an angle greater than 90°. The distance a^ve_1 may be laid off from e^h in the opposite direction if preferred (that is, on the same side of HQ as the projection a^h), representing the revolution of Q through an angle less than 90°. It is customary to place the revolved position where it will be as far from the other parts of the figure as possible.

In Fig. 218, the point a, lying in Q, is revolved about VQ into V. The construction is entirely analogous to that for revolving about HQ into H, and is left without further explanation to the student.

Working Rule. Returning to Fig. 217, the revolved position, a_r, is obtained by laying off from e^h, on the perpendicular

$a^h e^h$, a certain distance, namely, the true length of the line ae. The situation of this line is such that it is not necessary to draw the complete construction, or even the V-projection, $a^v e^v$, in order to find the true length. For, it will be seen that the true length, $a^v e_1$, is the hypothenuse of a right triangle, of which the base is $e_1 f = a^h e^h$, and the altitude $a^v f$. We may therefore state the following working rule: To revolve point a, lying in the plane Q, about HQ into H, proceed as follows. From a^h draw a line perpendicular to HQ. Then the revolved position, a_r, lies on this perpendicular, at a distance from HQ equal to the hypothenuse of a right triangle, of which the two sides are the distances a^h to HQ, and a^v to GL. This hypothenuse can be found directly with the dividers, as follows: Take the distance $a^h e^h$; set it off from f to point e_1 on the ground line; span the distance from e_1 to a^v.

Similarly, to revolve a point lying in a plane about the V-trace of the plane into V (Fig. 218), find the hypothenuse of a right triangle, of which the two sides are the distance from the V-projection of the point to the V-trace of the plane, and from the H-projection of the point to the ground line. This hypothenuse is the true distance of the revolved point from the V-trace of the plane.

SPECIAL CASE. The trace of the plane about which the revolution is made is perpendicular to the ground line. Let HQ, Fig. 219, be perpendicular to GL, VQ being at any angle. The plane Q is therefore perpendicular to V, so that the revolution of the given point occurs in a plane which is parallel to V. The path of the revolution must then project on V in its true shape, that is, as a circular arc, and the revolved position, a_r, can be obtained at once by projecting from the intersection of this arc and the ground line.

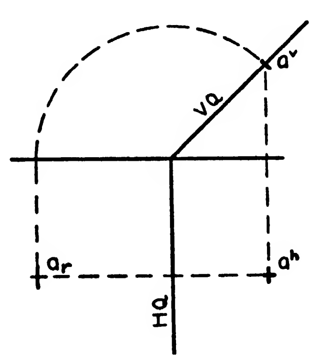

FIG. 219.

The ease with which the revolved position is obtained in this particular situation of the plane suggests another method for solving the general case, since an auxiliary plane of projection can always be assumed perpendicular to either trace of

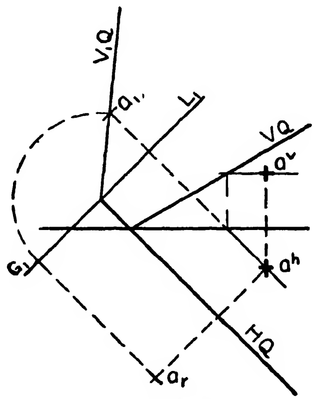

FIG. 220.

any given plane (see § 70). Such a construction is shown in Fig. 220, and is left without further explanation to the student.

COROLLARY. *To find the revolved position of a line lying in a plane, when the plane is revolved into H or V about the corresponding trace.*

Since a straight line is determined when two of its points are known, this problem can always be solved by revolving two points of the line by the method already given. However, a simpler method can usually be found, as in the following examples.

(1) Let the line A, Fig. 221, which is to be revolved about VQ into V, intersect VQ in the point t. Revolve any point of A, as $c(c^h, c^v)$ to c_r. If now we attempt to revolve the point t, this point, being on the axis of revolution, will not move. Hence the revolved position of the line, A_r, passes through c_r and t.

(2) Let the line A, Fig. 222, which is to be revolved about VQ into V, be parallel to VQ. Revolve any point c, of A, to c_r. Every other point of A is the same distance from the axis

VQ, and will revolve just as far. Hence, the revolved position A_r passes through c_r and is parallel to *VQ*.

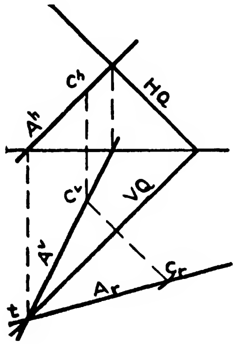

FIG. 221.

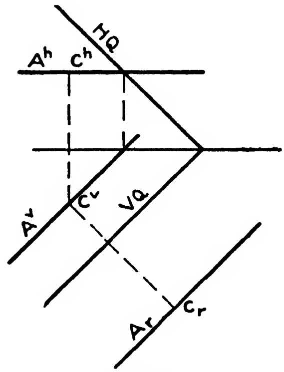

FIG. 222.

139. Solution of Plane Problems by Revolution of the Plane. By extending the preceding problem, any number of points or lines lying in a plane can be revolved into one of the coördinate planes, and the revolved position will show the true relation existing between them. The problem thus becomes a very useful auxiliary in the solution of a certain class of problems.

In revolving about either trace of a plane, as for example about *HQ* into *H*, it should be noticed that, while the *V*-projections of the points and lines concerned must be considered, no use is made of the *V*-trace of the plane, *VQ*. Similarly, when revolving about *VQ* into *V*, no use is made of *HQ*. Hence, when a plane is introduced as an auxiliary, for the express purpose of revolving about one of its traces, it is usually unnecessary to find more than one trace of the plane.

140. The Angle between Two Intersecting Lines. The angle between two intersecting straight lines is a plane angle. It is measured in the plane that is determined by the two lines.

Problem 22. ***To find the angle between two intersecting lines.***

Analysis. Find the plane containing the given lines. About one trace of this plane, revolve the lines into the corresponding coördinate plane, when the true angle between them will appear.

Construction (Fig. 223). Let A and B be the given lines. Find either trace of the plane containing these lines, for example the H-trace, HX (Prob. 6, § 106). Revolve both lines

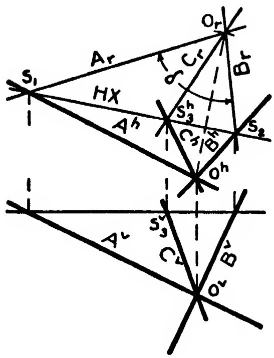

FIG. 223.

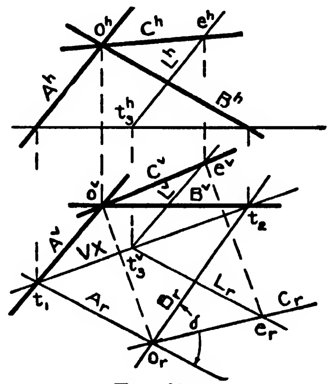

FIG. 224.

about HX into H. To accomplish this most readily, revolve their point of intersection, o, to o_r (Prob. 21, Working Rule, § 138). Then A_r passes through o_r and the trace s_1, while B_r passes through o_r and the trace s_2 (Corollary, § 138). The angle, δ, between A_r and B_r is the true angle between the lines.

A similar construction is shown in Fig. 224, in which the V-trace, VX, of the plane containing A and B is used. The angle, δ, between A_r and B_r, is the true angle between the lines.

SPECIAL CASE. One line is parallel to H, and the other is parallel to V. The angle may be found by the general method, but the following is simpler.

Analysis. Revolve either line about the other as axis, until both are parallel to the same coördinate plane.

Construction (Fig. 225). Let A and B be the given lines. Choose one of them, as A, for the axis. Since A is parallel to H, B must now be revolved about A until B, also, is parallel to H. Assume any point $c(c^h, c^v)$ in the line B, at a reasonable distance from the intersection, o, of the lines. Through c^h draw an indefinite line perpendicular to A^h. Take the distance o^vc^v in the compasses, and with o^h as center strike an arc across

this perpendicular, giving c_r. Then $o^h c_r$ is B revolved, B_r, and the angle between B_r and A^h is the true angle δ, between the given lines.

Proof of the Construction. Since A is parallel to H, the revolution of point c about A will project on H as a straight line perpendicular to A^h (§ 76). Hence c revolved, c_r, must lie somewhere on the perpendicular to A^h through c^h. Also, when the portion oc of the line B is revolved until parallel to H, it will appear in true length. Since the point o in the axis A does not move, $o^h c_r$ should equal the true length of oc. But $o^h c_r$ was made equal to $o^v c^v$, which is the true length of oc, since the line B is parallel to V. Hence the construction is correct.

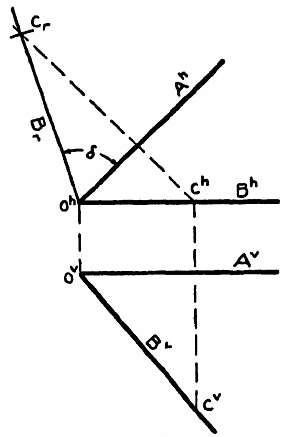

FIG. 225.

COROLLARY. *To find the projections of the bisector of the angle between two intersecting lines.*

Let it be required to find the bisector of the acute angle. In Fig. 223, draw first the actual bisector, C_r, in the revolved position, bisecting the angle between A_r and B_r. Now C_r necessarily passes through o_r, and in this case intersects the trace HX in the point $s_3{}^h$. Imagine the plane X to be counter-revolved (revolved back) about HX to its original position, taking with it the lines A, B, and C. The lines A and B, and their intersection, point o, will resume their former positions. The point $s_3{}^h$, being on the axis of revolution HX, will not move. Hence the H-projection, C^h, of the line C must pass through o^h and $s_3{}^h$.

To find the V-projection of C, note that o^h must be projected to o^v, while $s_3{}^h$, being an H-trace, lies in H, and must project to $s_3{}^v$ in GL. Hence C^v is determined by o^v and $s_3{}^v$.

The second example, Fig. 224, cannot be solved, however, in this way. As soon as the bisector, C_r, is drawn in the revolved position, it is seen that the intersection of C_r with the trace VX

is beyond the limits of the figure. One point of C^v is o^v, and one point of C^h is o^h. It remains to find a second point in each projection. Assume a point, $e_{,}$, in $C_{,}$. Through $e_{,}$ draw the revolved position, L_r, of a line lying in the plane of the given lines, by making L_r parallel to the revolved position of either of the given lines A or B; in this case L_r is drawn parallel to A_r. Since the line L is in the plane X, the intersection, t_3^v, of L_r and VX is the V-trace of the line L, and t_3^h is in GL. Also, since the line L is parallel to the line A, through t_3^v draw L^v parallel to A^v, and through t_3^h draw L^h parallel to A^h. Now e is a point in the line L. Hence revolve back from e_r perpendicular to VX, and find e^v on L^v; project from e^v and obtain e^h on L^h. Since e is a point also in the line C, we have C^v determined by o^v and e^v, and C^h determined by o^h and e^h.

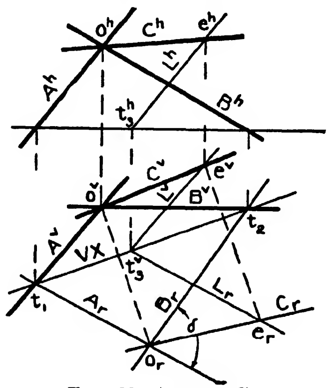

FIG. 224 (repeated).

141. The Angle between a Line and a Plane. The angle which a line makes with a plane is equal to the angle between the line and its projection on the plane. Hence we only need to find the angle between two intersecting straight lines.

Problem 23. ***To find the angle between a line and a plane.***

First Analysis. Project the given line on the given plane. Find the true angle between the given line and its projection; this is the angle required.

Second Analysis. From any point of the given line, drop a perpendicular to the given plane. Find the true angle between the given line and the perpendicular; this is the *complement* of the required angle.

In general, the second analysis gives the simpler construction, and is the one usually adopted.

Construction (Fig. 226). Let A be the given line and Q the given plane. Assume any point c (c^h, c^v) on A. From c draw

the line B (B^h, B^v) perpendicular to the plane Q (§ 111). Find the true angle, δ, between the lines A and B (Prob. 22, § 140). Then γ, which is the complement of δ, is equal to the true angle between line A and plane Q.

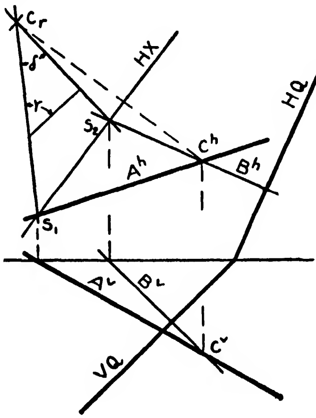

FIG. 226.

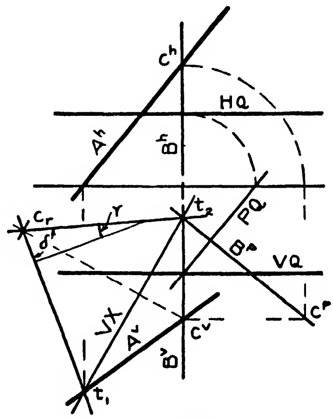

FIG. 227.

In Fig. 227, the given plane Q is parallel to the ground line. The general case still applies, but there is an extra detail in the construction. As before, assume any point c in the line A. From c drop the perpendicular B to the plane Q. Find the angle, δ, between the lines A and B. Then γ, the complement of δ, is the angle required. As a necessary step in finding the angle δ, we need one trace, as VX, of the plane containing the lines A and B. To determine VX we need the V-trace of each of these lines. The V-trace of A is readily found (Prob. 1, § 37). But B is a profile line, of which but one point, c, is known. Nevertheless, the line B is a definite line, since it is perpendicular to Q. To find the V-trace of B, proceed as follows: Find the profile projection, c^p, of c; also the profile trace, PQ, of Q. From c^p draw the profile projection of B, B^p perpendicular to PQ. Then the V-trace of B is the point t_2, in which B^p intersects VP. (See Prob. 11, Special Case 2, § 115.)

142. A Rectilinear Figure Lying in a Plane. A plane polygon of more than three sides cannot be assumed by assuming at random the projections of a number of points in space, and then connecting these points by straight lines. For, as soon as three points not in the same straight line have been assumed, a plane is determined; thereafter some consideration is necessary to assure us that the remaining points lie in the plane determined by these first three points.

Problem 24. ***Given one complete projection, and three points in the other projection of a plane polygon, to complete the projection.***

First Analysis. Lines which intersect, or are parallel, are necessarily in a plane. Hence, by producing the sides of the given complete projection until they intersect, or by drawing in this projection lines which are parallel to some of its sides or which intersect the sides or each other, auxiliary lines are obtained by means of which the second projection can be built up.

In particular, if the figure is a quadrilateral, the intersection of its diagonals is a point in the plane of the figure.

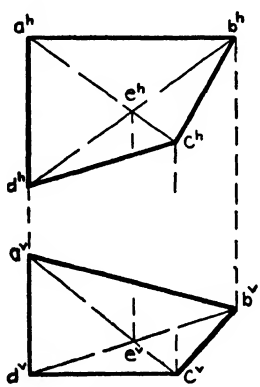

Fig. 228.

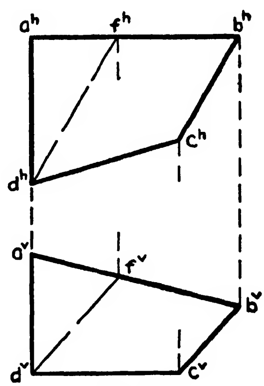

Fig. 229.

Second Analysis. Both projections of three points are known. Find the traces of the plane containing these points. For the remaining points, one projection of each point, and the traces of the plane containing it, are known; hence the second projection can be found.

Construction by First Analysis (Fig. 228). Let the H-projection, $a^hb^hc^hd^h$, and $a^vb^vc^v$ of the V-projection be given. Since this figure is a quadrilateral, we may draw the diagonals a^hc^h and b^hd^h, intersecting in the point e^h. In the V-projection draw a^vc^v. Locate e^v on this line by projecting from e^h. Draw b^ve^v, produce it, and locate on it d^v by projecting from d^h.

Otherwise (Fig. 229), through d^h draw d^hf^h parallel to b^hc^h, and intersecting a^hb^h at f^h. Project from f^h to f^v in a^vb^v. Through f^v draw f^vd^v parallel to b^vc^v. Then the line df, since it passes through the point f in ab, and is parallel to bc, lies in the plane of the quadrilateral. Locate d^v by projecting from d^h,

A second example is given in Fig. 230. Let the complete V-projection, $a^vb^vc^vd^ve^v$, and the H-projection of three points,

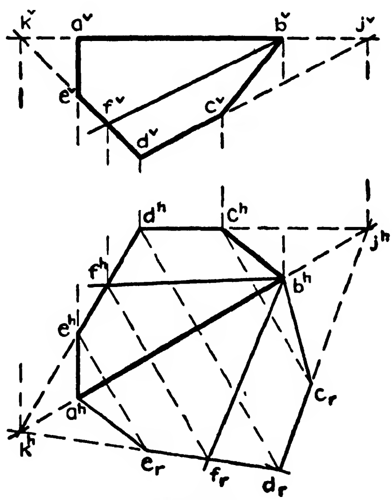

FIG. 230.

a^h, b^h, and c^h, be given. In this polygon, d^vc^v and d^ve^v can both be produced to meet a^vb^v produced in the accessible points j^v and k^v respectively. Produce a^hb^h, and project j^v and k^v on it. Draw j^hc^h. This line produced contains d^h, which is found by projection from d^v. The point d^h being thus located, draw d^hk^h, and project e^h from e^v. Draw e^ha^h, completing the polygon.

These constructions are given merely as illustrations of the

method. Other constructions should suggest themselves to the student. There is often a chance for the exercise of considerable ingenuity in solving problems in this manner, especially in the matter of obtaining accuracy in the result. Note that the ground line is not essential in these constructions.

Construction by Second Analysis (Fig. 231). Let the complete V-projection, $a^v b^v c^v d^v e^v$, and three points, a^h, b^h, and c^h, of the H-projection be given. By producing the lines ab and bc, obtain the plane X, containing these two intersecting lines (Prob. 6, § 106). Points d and e lie in this plane, and d^v and and e^v are known; find d^h and e^h (Prob. 16, § 133). Connect $c^h d^h$, $d^h e^h$, $e^h a^h$.

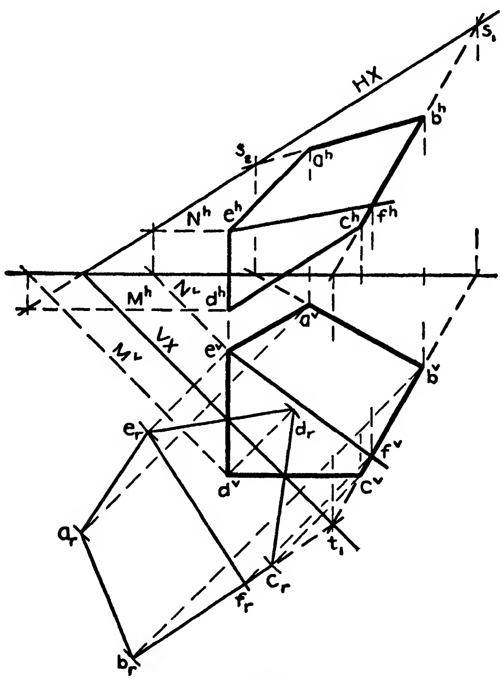

FIG. 231.

COROLLARY 1. *To find the true size and shape of a plane polygon.*

First Analysis. Find one trace of the plane containing the

polygon. Revolve the polygon about this trace into the corresponding coördinate plane.

Second Analysis. If any side of the polygon is parallel to one of the coördinate planes, the polygon may be revolved about this line as an axis into a position parallel to this coördinate plane.

Construction by First Analysis (Fig. 231). Each corner of the polygon is here revolved about VX into V (Prob. 21, Working Rule, § 138), due regard being had to the relative positions of the points with respect to the axis VX.

Check. Note that, for each side of the polygon, the V-projection and the revolved position (produced if necessary) must intersect VX at the same point, namely, the V-trace of this line.

Construction by Second Analysis (Fig. 230). In this polygon, the line ab is parallel to H. Hence the polygon may be revolved about a^hb^h into a position parallel to H. Revolve first the point d. To do this, through d^h draw a line perpendicular to a^hb^h. Obtain the true length of the line dj (Prob. 3, § 78). In this case d^vj^v is the true length, since dj is parallel to V. With j^h as center, radius equal to the true length of dj, strike an arc across the perpendicular from d^h, thus obtaining d_r. From d_r draw to the fixed points j^h and k^h on the axis a^hb^h. Obtain c_r on d_rj^h, and e_r on d_rk^h, by drawing from c^h and e^h perpendicular to a^hb^h. Then since a^h and b^h are fixed points, the true size of the polygon is $a^hb^hc_rd_re_r$.

Corollary 2. *To find the projection of the line which bisects one of the interior angles of a plane polygon.*

Construction. In Fig. 231 the interior angle at e is bisected. The bisector is first drawn in the revolved position, and intersects the side b_rc_r in the point f_r. Project from f_r perpendicular to VX to f^v in b^vc^v. and from f^v perpendicular to GL to f^h in b^hc^h; then e^vf^v and e^hf^h are the projections required.

In Fig. 230, the interior angle at b is bisected, the actual bisector, b^hf_r, intersecting d_re_r in the point f_r. From f_r are obtained f^h and f^v, giving b^hf^h and b^vf^v as the required projections.

CHAPTER XVI

MISCELLANEOUS PROBLEMS ON THE LINE AND PLANE

143. The Distance from a Point to a Line. The shortest distance from a point to a given straight line is obtained by dropping a perpendicular from the point to the line. This perpendicular evidently lies in the plane which is determined by the given point and line.

Problem 25. ***To find the shortest (perpendicular) distance from a point to a line.***

Analysis. In the general position of the line, the right angle between it and the perpendicular will not project as such (§ 109). Hence, pass an auxiliary plane through the given line and point. Revolve this plane into either coördinate

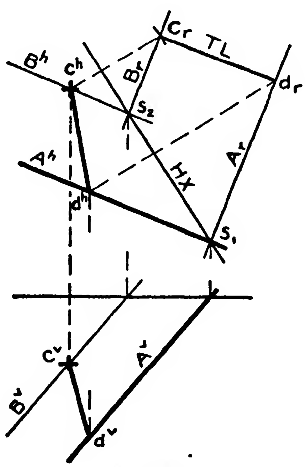

FIG. 232.

plane, obtaining thus the true relative position of the line and point. Draw the required perpendicular in the revolved position. To find the projections of the perpendicular, revolve the auxiliary plane back to its original position.

Construction (Fig. 232). Let A be the given line, and c the given point. Through c draw the auxiliary line $B\,(B^h, B^v)$ parallel to A (§ 93). Pass the auxiliary plane X through the lines A and B (Prob. 6, § 106). Since X is introduced solely for the purpose of obtaining a revolved position, but one trace, as HX, is necessary (§ 139). Revolve A and c about HX into H. To do this most readily, revolve point c to c_r (Prob. 21, Working Rule, § 138); draw B_r through c_r and the trace s_2 (Prob. 21, Corollary, § 138), then draw A_r through the trace s_1 and parallel to B_r. From c_r drop the perpendicular c_rd_r to A_r; this is the actual shortest distance required. To find the projections of the perpendicular cd, revolve back from d_r perpendicular to HX, and find d^h in A^h; draw c^hd^h. Project from d^h to d^v in A^v, and draw c^vd^v.

Special Case I. The given line is parallel to one of the coördinate planes (Fig. 233). Let the given line A be parallel to H, and let c be the given point. It is required to draw from

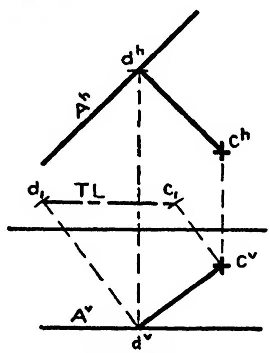

Fig. 233.

c a line which shall be perpendicular to the given line A. Now if two lines are at right angles, the right angle will project as such if one of the lines is parallel to a coördinate plane (§ 110). Since A is parallel to H, draw c^hd^h perpendicular to A^h, giving at once the H-projection of the perpendicular from c to A. Project from d^h to d^v in A^v; draw c^vd^v. For the actual distance from the point to the line, find the true length of cd (Prob. 3, § 80).

The above solution suggests a method for solving the general case, since any line can be converted into a V-parallel by assuming a secondary plane of projection parallel to the H-projection of the line (§ 70). This is illustrated in Fig. 234. In that figure, ab is the given line, c the given point, and a secondary projection is made after assuming G_1L_1 parallel to

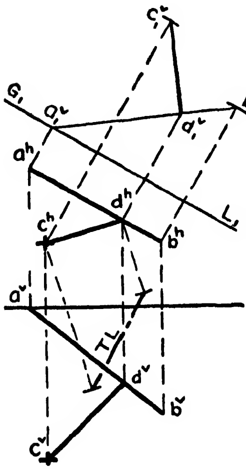

Fig. 234.

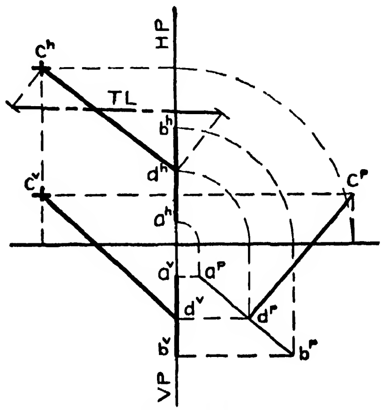

Fig. 235.

a^hb^h. Further explanation of the construction is omitted, since, for the line in question, the solution is of doubtful advantage as compared with the general solution.

If, however, the given line is a profile line, the simplest solution is obtained by the use of an additional projection.

Special Case II. The given line lies in a profile plane. Let ab, Fig. 235, be the given line, and c the given point. Find the profile projection, a^pb^p, of the line ab. Project c on to P, and find the profile projection c^p. From c^p draw, perpendicular to a^pb^p, the profile projection, c^pd^p, of the required perpendicular. From d^p find the projections d^v and d^h; then c^hd^h and c^vd^v are the required projections. The true length of cd (Prob. 3, § 80) is the required shortest distance.

144. Distance between Two Lines. If two lines are not in the same plane, that is, neither intersect nor are parallel, there is but one line that is perpendicular to both of them. This line, the common perpendicular, measures the shortest distance between the two given lines.

Problem 26. ***To find the shortest distance between two lines not in the same plane, and the projections of their common perpendicular.***

Analysis (Fig. 236). Let A and B be the given lines, in any oblique position in space. Through either given line, as A, pass an auxiliary plane Q, parallel to the other given line B. Then B is everywhere equally distant from Q, and

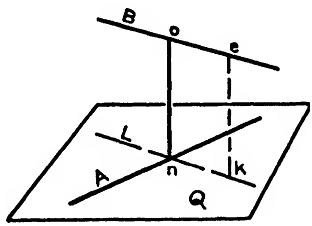

FIG. 236.

this distance is equal to the shortest distance between B and A. Assume any point, e, in B; from e drop a perpendicular to the plane Q, intersecting Q at the point k. Note that although this first perpendicular ek is not in the required position, it does have the required direction. Now let the perpendicular ek be moved, parallel to itself, so that the point e moves in the line B. The point k will move in the plane Q, describing in this plane a line L, parallel to B. Since A is also in Q, and since L is not parallel to A because B is not, L and A will intersect in a point, n. Let the perpendicular ek come to rest when point k coincides with point n. The line will then be in the position on, intersecting both A and B and perpendicular to each. Hence on is the common perpendicular required. The actual distance between the lines A and B may be found by measuring the length of either ek or on.

Construction (Fig. 237). Let A and B be the given lines. Through A pass the plane Q, parallel to B (Prob. 7, § 107). To do this, assume any point, c, in A. Through c draw line D parallel to B. Pass the plane Q through the lines A and D. Next assume any point e in B. From e draw the line F perpendicular to Q (§ 111). Find the point k, in which F intersects Q (Prob. 13, § 119). Here the auxiliary plane X perpendicular to H is used. Planes X and Q intersect in the line J; J^v and F^v intersect in k^v, one projection of the required point k.

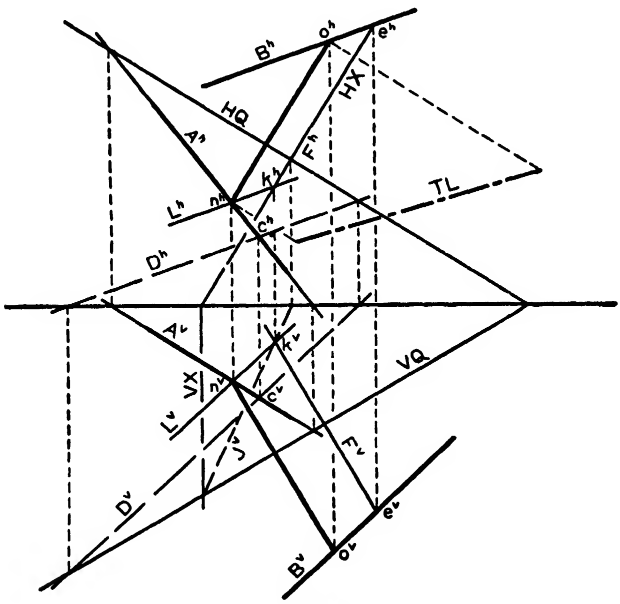

FIG. 237.

Let now the trial perpendicular ek be carried parallel to itself to the position on. In this translation, point e moves along the line B, and point k moves in the line L, parallel to B, to the point n, where L intersects A. Consider each projection separately. In the H-projection, through k^h draw L^h parallel to B^h; L^h intersects A^h at n^h; through n^h draw n^ho^h parallel to e^hk^h to intersect B^h at o^h. Make the analogous construction in the V-projection. Thus n^v and o^v are determined independently

of n^h and o^h. But n^h and n^v are two projections of the same point; likewise o^h and o^v. Hence, as a check on the construction, n^h and n^v should lie in the same projector, as should also o^h and o^v.

The line no (n^ho^h, n^vo^v) is the required common perpendicular. The shortest distance between the given lines is equal to the true length of no (Prob. 3, § 80).

SPECIAL CASE I. One of the given lines is parallel to H or V. The general solution will apply to this case; but a particular solution, much simpler than the general one, can be had by using a secondary plane of projection.

Let A and B, Fig. 238, be the given lines, the line A being parallel to H. Assume a secondary ground line G_1L_1, perpen-

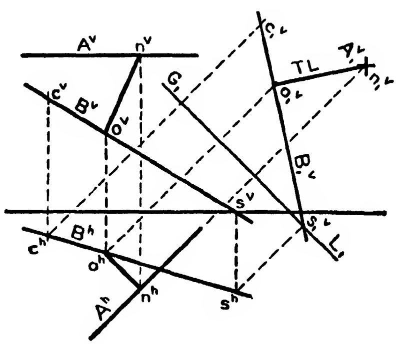

FIG. 238.

dicular to A^h. Project the given lines A and B on the secondary vertical plane V_1 thus assumed (§ 68). The line A will project as a point, A_1^v. The required line no is perpendicular to both A and B. Since A is perpendicular to V_1, no, which is perpendicular to A, is parallel to V_1. Again, since no and B are perpendicular, and no is parallel to V_1, the projection $n_1^vo_1^v$ and B_1^v are perpendicular (§ 110). Hence, from A_1^v draw $n_1^vo_1^v$ perpendicular to B_1^v. Project from o_1^v on B_1^v to o^h on B^h, thence to o^v on B^v. From o^h draw o^hn^h perpendicular to A^h (see Prob. 25, Special Case I, § 143). Project from n^h to n^v in A^v, and draw o^vn^v. Then n^ho^h and n^vo^v are the projections of the common perpendicular, while the actual shortest distance is equal to $n_1^vo_1^v$.

Special Case II (Fig. 239). The given lines, ab and cd, are both profile lines. Project both lines on to the same profile plane; the plane here used is the one containing the given line ab. Find the profile projections, a^pb^p and c^pd^p. Then it is evident that the common perpendicular, ef, will project on P as a point, located at the intersection of a^pb^p and c^pd^p. The rest should be obvious.

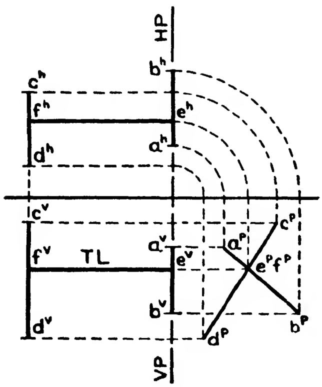

Fig. 239.

Note. If one of the given lines is a profile line, the other being a general oblique line, the general solution applies. The simpler construction results when the auxiliary plane Q is passed through the profile line.

145. The Angle between Two Planes. Let two planes intersect so as to form a dihedral angle. The figure is left for the student to draw. The angle between these planes is measured by two lines, one in each plane, each perpendicular at the same point to the line of intersection of the given planes. It is evident that these two lines lie in a third plane which is perpendicular to both of the given planes and to their line of intersection. They are, in fact, the lines of intersection of this third plane with the given planes.

Problem 27. ***To find the angle between two planes.***

First Analysis. From any point in space, drop a perpendicular to each of the given planes. The angle between these lines is equal to the angle between the planes.

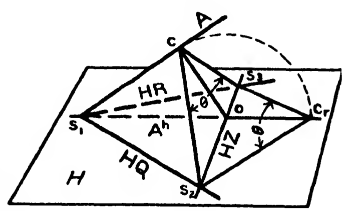

Fig. 240.

Second Analysis. Let Q and R, Fig. 240, be the given planes, and let these planes intersect in the line A. At any point c in A, pass an auxiliary plane Z, perpendicular to A. The plane Z intersects Q in the line cs_2 and R in the line cs_3. The angle

between cs_2 and cs_3 is the required angle between Q and R. Let A^h be the projection of A on H. Since Z is perpendicular to A, A^h and HZ are perpendicular (§ 111). The projection A^h and the trace HZ intersect in the point o; connect c and o. Now co is perpendicular to A, since it lies in the plane Z which is perpendicular to A. Also, co is perpendicular to HZ, since it lies in the plane of the lines A and A^h, which is perpendicular to HZ. Revolve the plane Z about HZ into H. Since co is perpendicular to HZ, c will revolve with o as a center, and will fall on A^h, because the latter is also perpendicular to HZ, at the distance $c_ro = co$. The lines cs_2 and cs_3 will take the revolved positions c_rs_2 and c_rs_3, and will show the true size of the angle θ, between them.

In actual construction there is little choice between the first and the second analysis. Moreover, the latter usually adapts itself better to the way in which the problem most often occurs in practice. Hence the constructions here given will be made in accordance with the second analysis.

Construction (Fig. 241). Let Q and R be the given planes. Find the line of intersection, A, of these planes (Prob. 12 § 118). Assume an auxiliary plane, Z, perpendicular to A, by drawing HZ perpendicular to A^h. The V-trace, VZ, is not needed (§ 139). The trace HZ intersects A^h in the point o, lying in H. It is necessary to find the shortest distance from this point to the line A. Assume a secondary ground line, G_1L_1, coincident with A^h. Project the line A as A_1^v (§ 68), using the traces s and t as the most convenient points in the line. Point o remains in G_1L_1. From o draw oc_1^v perpendicular to A_1^v; this is the shortest distance from o to A. On A^h, lay off $oc_r = oc_1^v$, thus obtaining c_r. Note the points s_2 and s_3, in which HZ intersects HQ and HR

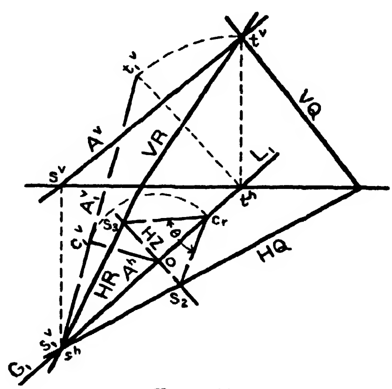

FIG. 241.

respectively. Draw $c_r s_2$ and $c_r s_3$. The angle between these lines equals the required angle, θ, between the given planes.

Additional examples are given in Figs. 242 and 243. Each of them falls under the general case, but they differ in the method of finding the line of intersection of the planes.

In Fig. 242 the line A passes through the point s_1, in which HQ, HR, VQ and VR intersect. A second point is

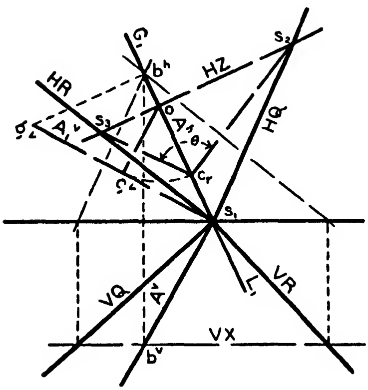

FIG. 242.

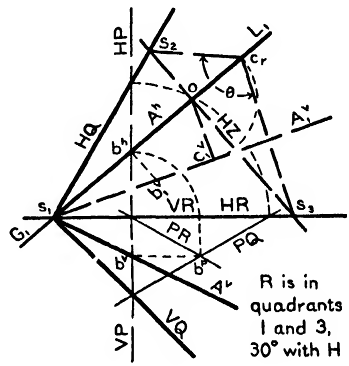

FIG. 243.

obtained by means of the auxiliary plane X (Prob. 12, Special Case 5, § 118). The secondary projection A_1 passes through s_1 and the secondary projection of b.

Figure 243 is similar to Figure 242 in that the traces HQ, HR, VQ, and VR intersect in a common point s_1. A second point b, in the line of intersection A, is here obtained by means of the profile plane P (Prob. 12, Special Case 6, § 118), and the secondary projection $A_1{}^v$ passes through s_1 and the secondary projection, $b_1{}^v$, of b. In other respects the constructions of Figs. 242 and 243 follow that of Fig. 241.

SPECIAL CASE I. The line of intersection of the given planes is parallel to H or V. Let Q and R, Fig. 244, be the given planes, with HQ and HR parallel. Then the line of intersection, A, is parallel to H (Prob. 12, Special Case 1, § 118). Pass the auxiliary plane Z perpendicular to A. Since

A is parallel to *H*, the plane *Z* is perpendicular to *H*. The line *A* intersects the plane *Z* in the point $c\,(c^h, c^v)$ (Prob. 13, Special Case, § 119). Revolve the plane *Z* about *HZ* into *H*. The revolved position, c_r, is obtained simply by making the distance from *HZ* to c_r equal to the distance of c^v from *GL*. Then it is evident that the lines of intersection of *Z* with the given planes *Q* and *R* will take the positions c_rs_2 and c_rs_3, and

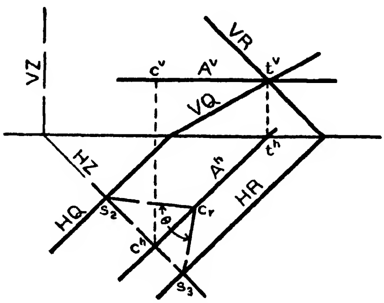

FIG. 244.

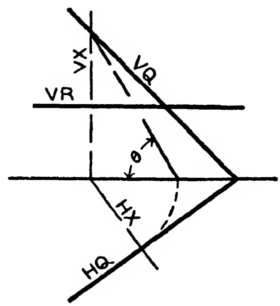

FIG. 245.

the included angle, θ, between these lines is the angle between the given planes.

NOTE. The student may discover that *VZ*, and also c^v, are superfluous in this construction.

SPECIAL CASE II. One of the given planes is parallel to *H* or *V*. This case falls under the solution just given; a still simpler solution is shown in Fig. 245. The given plane *R* is parallel to *H*. Therefore the angle between planes *Q* and *R* is equal to the angle, θ, which *Q* makes with *H*, the latter angle being found by Problem 18, § 136.

146. Application of the Preceding Problem. As instances in which the angle between two planes is wanted in actual construction, we may cite the case of two intersecting pitch (sloping) roofs, or of two intersecting masonry walls with battered (sloping) faces. The way in which the problem usually appears is as a corollary to the following problem.

Problem 28. *Given the horizontal traces of two planes, and the angle each plane makes with H, to find the line of intersection of the two planes.*

NOTE. For the problem as stated, four results are possible, in general. In addition to the above data, it will be assumed that the slopes of the planes are known in order to limit the problem to one result. This would be the case in the practical applications of the problem.

Analysis. One point in the line of intersection is located at the intersection of the given traces of the planes. To locate a second point, draw in each plane a line parallel to H, so that these lines will intersect. The necessary and sufficient condition for their intersection is that these two lines shall be at the same distance from H.

Construction (Fig. 246). Let HQ and HR be the given H-traces. Let plane Q make the angle α_1 with H, and R the angle α_2 with H. In any convenient position, draw a ground line,

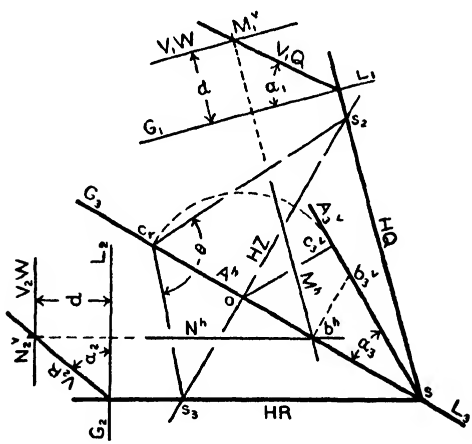

FIG. 246.

G_1L_1, perpendicular to HQ, and draw the V-trace and edge view, V_1Q, of Q, making the given angle, α_1, with G_1L_1. Draw a second ground line, G_2L_2, perpendicular to HR, and draw the V-trace and edge view, V_2R, of R, making the given angle, α_2, with G_2L_2. Assume any arbitrary distance, d. Draw the

V-trace and edge view, V_1W, of a plane W, parallel to H and at the distance d above H, also the V-trace and edge view, V_2W, of this same plane, that is, draw V_1W and V_2W parallel respectively to G_1L_1 and G_2L_2, and at the assumed distance d. The planes Q and W intersect in a line M, one projection of which is the point, M_1^v, where V_1W and V_1Q intersect. Project from M_1^v and obtain M^h, parallel to HQ. The planes R and W intersect in a line N, a projection of which is the point N_2^v, where V_2W intersects V_2R. Project from N_2^v and obtain N^h, parallel to HR. The projections M^h and N^h intersect in b^h, which is an actual intersection in space, since M and N are both in the plane W. This point is common to both Q and R. Hence the H-projection, A^h, of the required intersection of Q and R, is drawn through b^h and the intersection, s, of HQ and HR.

To locate the line A further, assume a third ground line, G_3L_3, coincident with A^h. A V-projection, b_3^v, of point b, is located by making the distance from b_3^v to G_3L_3 equal to the distance d, since this is the distance from H of any point in the lines M and N. The projection, A_3^v, of A passes through b_3^v and the point s. The line A is now definite, since we have its H-projection, A^h, and the angle α_3 (between A_3^v and G_3L_3) which this line makes with H.

COROLLARY. *To find the angle between the given planes.*

In finding the angle between two planes by Problem 27, the construction already made in Fig. 246 is a part of the construction of that problem; namely, the H-projection, A^h, of the line of intersection of the given plane, and the projection, A_3^v, of this line, using a secondary ground line coincident with A^h. To complete the construction, draw, through any convenient point o on A^h, the trace HZ perpendicular to A^h. Find the perpendicular distance, oc_3^v, from o to A_3^v. Locate c_r on A^h by making the distance c_ro equal to oc_3^v. From c_r draw to the intersections, s_2 and s_3, of HZ with HQ and HR respectively. The angle, θ, between these lines is the required angle between the planes Q and R.

CHAPTER XVII

COUNTER-REVOLUTION OF PLANES

147. Counter-revolution. A point, line, or any number of lines lying in a plane may be revolved about one of the traces of the plane into H or V. The basis for all such revolutions is the revolution of a single point, given in Problem 21, § 138.

It has further been shown (see Prob. 22, Corollary, § 140; Prob. 24, Corollary 2, § 142; Prob. 25, § 143) that the revolved position of a line lying in a plane can be given or assumed, and the plane *counter-revolved,* that is, revolved back about its trace so that the projections of the line can be found.

In all these solutions, however, the counter-revolution was made to depend upon a previous direct revolution of lines already existing in the plane. We shall now take up the problem of a direct counter-revolution, starting only with the revolved position of a point and the traces of the plane; in other words, the direct reverse of Problem 21, § 138.

Problem 29. ***Given the position of a point lying in a plane after the plane has been revolved into H or V about the corresponding trace, to find the projections of the point.***

Analysis. Suppose the point a, lying in the plane Q, to have been revolved about HQ into H. The figure is left to the student. The path of the revolving point is the arc of a circle, which will project on H as a straight line perpendicular to HQ, and on any plane perpendicular to HQ in its true shape and size. Referring to the various solutions of Problem 21, § 138, it is seen that the solution given in Fig. 220 shows these two projections of the path of the revolving point. The construction of Fig. 220 may therefore be reversed to give the solution of the present problem.

Construction (Fig. 247). The plane Q is given by its traces HQ and VQ, and a_r is given as the position of a point which

has been revolved about HQ into H. Assume a secondary ground line G_1L_1 perpendicular to HQ. Assume a point, t^v, in VQ; project to t^h in GL. Find the secondary projection, t_1^v, of t, and through this point draw the trace and edge view V_1Q (§ 70). Project a_r to G_1L_1. With o as center, revolve this

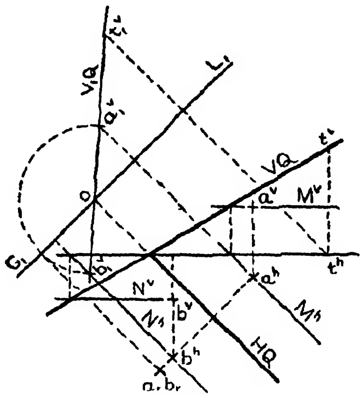

FIG. 247.

point to a_1^v in V_1Q. Then a_1^v is the secondary projection of the point a. Project from a_1^v perpendicular to G_1L_1, and from a_r perpendicular to HQ. Both of these lines must pass through a^h, which is thus determined. To find a^v, we now have given a^h as one projection of a point lying in the plane Q. Through a^h draw M^h parallel to HQ; this is one projection of a horizontal principal line of Q. The other projection, M^v, is parallel to GL; a^v lies in M^v (Prob. 16, § 133).

Check. The distance of a^v from GL equals the distance of a_1^v from G_1L_1 (§ 67).

A Second Result. The point a, thus found, lies above H. The given revolved position may also be the revolved position of a point b, lying below H. To obtain this result, after projecting from the given revolved position to G_1L_1, revolve about o to b_1^v, which lies in V_1Q produced below G_1L_1. Then proceed as for the point a. As before, there is a check on the construction; the distance from b^v to GL equals the distance from b_1^v to G_1L_1.

NOTE. The student does not always see readily why the V-projections a^v and b^v should be located by means of auxiliary lines in the plane Q, since the distances of these points from GL appear at once in the secondary projection. Indeed, the projections a^v and b^v can be located by transferring from the secondary projection the distances of $a_1{}^v$ and $b_1{}^v$ from G_1L_1. But if this method be employed, not only the distances, but the directions of these points from G_1L_1 must be considered, and there is a chance for error. The method of locating a^v and b^v by means of the auxiliary lines M and N is free from this ambiguity.

In Fig. 248, the given revolved position, a_r, is that of a point which has been revolved about VQ into V. A secondary

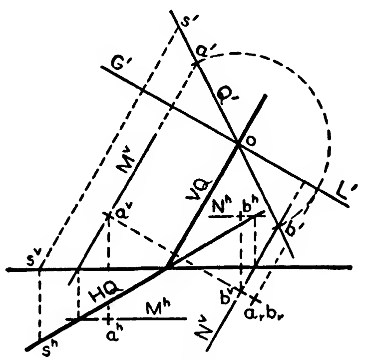

FIG. 248.

ground line, $G'L'$, is taken perpendicular to VQ, and the construction is entirely analogous to that of Fig. 247. Two results are possible, namely, points a (a^v, a^h) and b (b^v, b^h).

COROLLARY. *To counter-revolve a plane polygon.*

Analysis. Find the projections of each corner of the polygon according to the preceding method.

Construction (Fig. 249). The given revolved position, $1_r2_r3_r4_r$, is that of a square which has been revolved about HQ into H. As in the case of a single point, this may be the revolved position of two figures lying in Q. To limit the construction to one result, therefore, it is given that point 1 is the highest corner of the square. The projections of each corner of the square are obtained by the method of Fig. 247, using a secondary ground line perpendicular to HQ. Note especially point 3.

Since the edges of the square are lines lying in the plane *Q*, the traces of these lines must lie in the traces of the plane (§ 96). This fact furnishes a series of checks on the work, as, for example, that 2_r3_r and 2^h3^h intersect *HQ* in the same point,

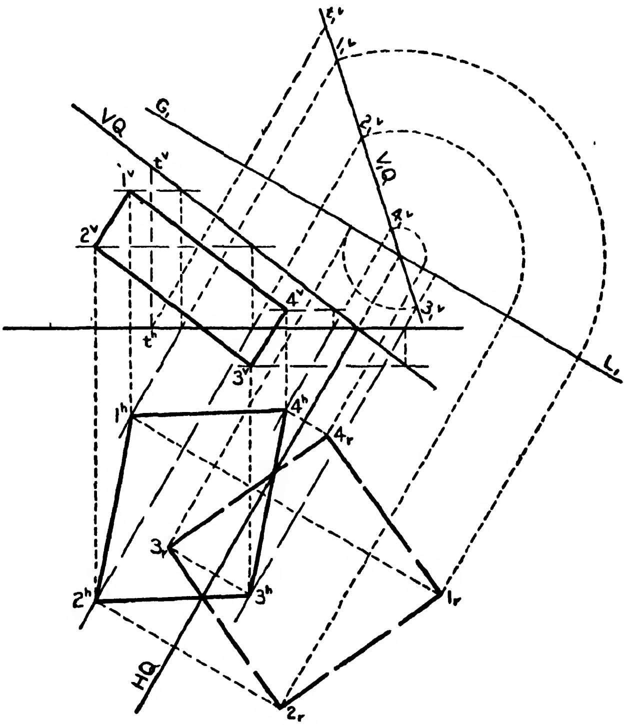

FIG. 249.

namely, the *H*-trace of this line, and that this trace projects to *GL* in the point where 2^v3^v intersects *GL*.

Similarly, 1_r4_r and 1^h4^h, if produced, must intersect *HQ* in the same point. In fact, the counter-revolution may be effected largely by the use of the traces of the various lines, but at least one point must be counter-revolved as in the general problem.

The counter-revolution of a plane figure is the reverse of Problem 24, Corollary 1, § 142, and should be compared with it.

148. An Auxiliary Problem. As an application of the preceding corollary, we shall construct the projections of a right prism or a pyramid resting on an inclined plane, using the polygon which is located in the plane as the base of the solid; or, what will result in the same construction, we shall construct the projections of a right prism or pyramid, the axis of which is inclined to both *H* and *V*. As, however, the axis of the pyramid, or the long edges of the prism, will be a line or lines perpendicular to the plane of the base, it will be necessary first to solve the following problem.

Problem 30. ***At a given point in a plane, to draw a line which shall be perpendicular to the plane and of given length.***

First Analysis. A line perpendicular to a plane and indefinite in length may be drawn by making the projections of the line perpendicular to the traces of the plane (§ 112). Through the given point draw a line perpendicular to the plane. Assume a second point on this line. Find the true length of the line between the assumed and given points. On this true length, measure from the given point the length of line desired. Obtain the projections of this line by reversing the construction for finding the true length.

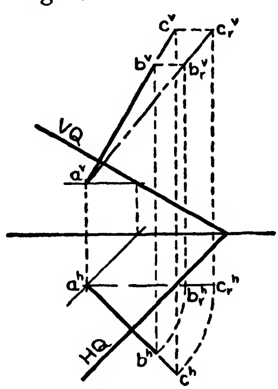

FIG. 250.

Construction by First Analysis (Fig. 250). Let a, lying in Q, be the given point. Draw the line ac (a^hc^h, a^vc^v), indefinite in length and perpendicular to Q. Assume a point c (c^h, c^v) on

this line, and find the true length, $a^v c_r^v$, of ac (Prob. 3, § 78). Measure off $a^v b_r^v$ equal to the true length of the required line. From b_r^v find the projections b^v and b^h by reversing the construction for finding the true length. Then $a^h b^h$ and $a^v b^v$ are the projections of a line of the given length.

NOTE. Any method of finding the true length of ac may be reversed, all leading to the same point b.

Second Analysis. Any plane perpendicular to the given plane will be parallel to the required line. Hence assume a secondary plane of projection perpendicular to either trace of the given plane. Find the secondary projection of the given plane and point on it. Since the required line is parallel to this plane of projection, the secondary projection of the line may be drawn at once, perpendicular to the secondary trace of the plane and of the given length. From the secondary projection of the line the projections on H and V may be found.

Construction by Second Analysis (Fig. 251). Let a, lying in Q, be the given point. Assume G_1L_1 perpendicular to HQ.

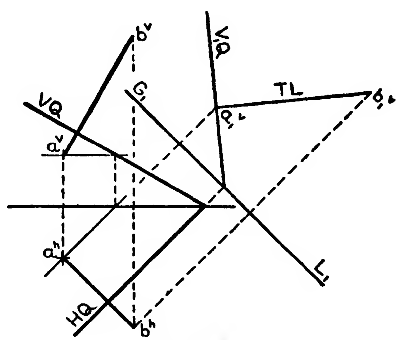

FIG. 251.

Project a to a_1^v, and through this point draw the V_1-trace and edge view, V_1Q. Draw $a_1^v b_1^v$ perpendicular to V_1Q, and make it equal to the given true length. Draw $a^h b^h$ perpendicular to HQ; locate b^h by projection from b_1^v. Project from b^h to b^v, making the distance from b^v to GL equal to the distance from b_1^v to G_1L_1. Draw $a^v b^v$. As a check, this should be perpendicular to VQ. Then $a^h b^h$ and $a^v b^v$ are the required projections.

149. A Prism or Pyramid Whose Axis Is Inclined to Both *H* and *V*. The method of obtaining the projections may best be shown by considering concrete examples.

EXAMPLE 1. *Draw the projections of a right pyramid.*

The axis is $1\frac{7}{8}''$ *long, makes* 45° *with H and* 30° *with V, and slopes (from the apex of the pyramid) downward, backward, to the right.*

The base is a regular pentagon, inscribed in a circle of 1″ *radius.*

The lowest edge of the base is parallel to H. The center of the base is $\frac{3}{4}''$ *above H and* 1″ *in front of V.*

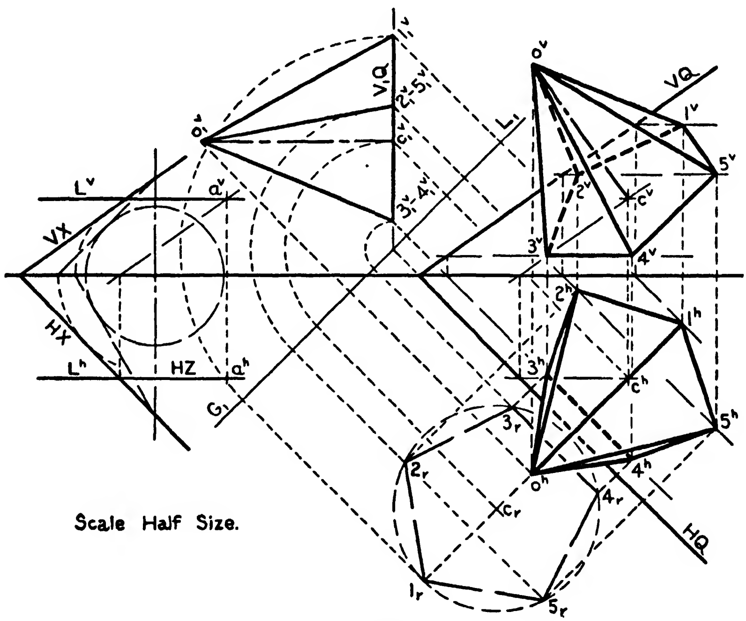

FIG. 252.

Construction (Fig. 252). The first step in the construction is to find the plane which contains the base of the pyramid. The data for this plane are not given directly in the statement of the problem. But, since the plane of the base of the pyramid is perpendicular to its axis, we derive at once, from

the data given for the axis, that this plane makes the complementary angles, 45° with H and 60° with V, and slopes downward, forward, to the left. Construct the plane X, using these angles and slope (Prob. 20, § 137).

We might next find a point, a, in this plane, distant $\frac{3}{4}''$ above H and $1''$ in front of V. To do this, draw the line L, every point of which is so located; that is, draw L^v parallel to GL and $\frac{3}{4}''$ above it, and L^h parallel to and $1''$ in front of GL. Find the point, a, where L pierces X (Prob. 13, § 119).

In the figure, however, in order to render more clear the subsequent steps of the construction, point c (c^h, c^v) is located at the given distances from H and V, and through this point the plane Q is passed parallel to X (Prob. 9, § 107).

Revolve c about HQ into H, obtaining c_r (Prob. 21, § 138). With c_r as center, draw a circle of $1''$ radius. Inscribe in this circle the regular pentagon $1_r2_r3_r4_r5_r$, so located that the side nearest HQ is parallel to it. This pentagon is the revolved position of the base of the pyramid.

Assume a secondary ground line, G_1L_1, perpendicular to HQ. Project c to c_1^v, and draw V_1Q through c_1^v. Using the edge view V_1Q, counter-revolve the pentagon into the plane Q (Prob. 29, Corollary, § 147).

To obtain the apex of the pyramid, we must draw from c a line $1\frac{7}{8}''$ long and perpendicular to Q. Draw from c_1^v the line $c_1^vo_1^v$, perpendicular to V_1Q; make the length of this line $1\frac{7}{8}''$.

Draw from c^h perpendicular to HQ; project from o_1^v to o^h on this line.

Project from o^h to o^v, making the distances of o^v and o_1^v from their respective ground lines equal. Note as a check whether o^vc^v is perpendicular to VQ. (Compare Fig. 251.)

Complete the projections of the pyramid, making visible edges full, and invisible ones dotted.

In the figure, the projection of the entire pyramid on the secondary plane of projection is completed, as well as the projections on H and V.

EXAMPLE 2. *Draw the projections of a right prism, resting with its lower base in a given plane Q. The prism is $1\frac{3}{4}''$ long. The base is a square of $1''$ sides. The lowest corner of the base is $\frac{1}{2}''$ above H, $\frac{3}{4}''$ in front of V. Two edges of the base make equal angles with H and V, and slope downward, backward, to the right.*

NOTE. A plane, like W, Fig. 253, which has coincident traces parallel to GL, will slope downward and backward, and will evidently make equal angles, namely, 45°, with H and V. Any line lying in such a plane will make equal angles with H and V.

Construction. Let Q, Fig. 254, be the given plane on which the base of the prism is to rest. Draw the line L (L^v, L^h), $\frac{1}{2}''$

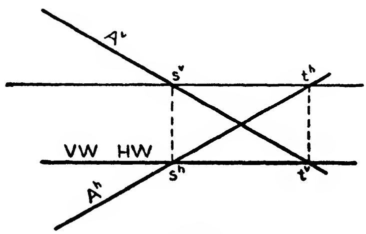

FIG. 253.

above H and $\frac{3}{4}''$ in front of V. Find the point in which the line L intersects the plane Q (Prob. 13, § 119). This locates the point 1, the lowest corner of the base. Pass through the line L a plane sloping downward and backward, and making 45° with H and V; since the line L is $\frac{1}{2}''$ above H, the trace, HW, is drawn $\frac{1}{2}''$ behind L^h. Find the line of intersection of the planes Q and W. This is $s1$, and is determined by the point s, where HW intersects HQ, and the point 1 already found.

The line $s1$ lies in Q, makes equal angles with H and V, and slopes downward, backward, to the right. Hence $s1$ determines one side of the base of the prism. Revolve $s1$ about HQ into H to the position s^h1_r (Prob. 21, Corollary, § 138). With s^h1_r (produced) as one side, draw the square $1_r2_r3_r4_r$, making each side $1''$ long; this square is the revolved position of the base of the prism. Note that, since point 1 is to be the lowest corner, no position of the square, other than that shown, is possible.

Assume G_1L_1 perpendicular to HQ; locate V_1Q by means of some point, as e, and counter-revolve the square (Prob. 29, Corollary, § 147). As a check, note that the point 2 must fall on the line $s1$, produced, already determined.

The long edges of the prism are lines $1\frac{3}{4}''$ long and perpendicular to Q. Draw these lines, $1_1{}^v 5_1{}^v$, $2_1{}^v 6_1{}^v$, etc., in the secondary projection; from this projection find the projections on H and V (Prob. 30, Second Analysis, § 148). Complete the projections of the prism as shown.

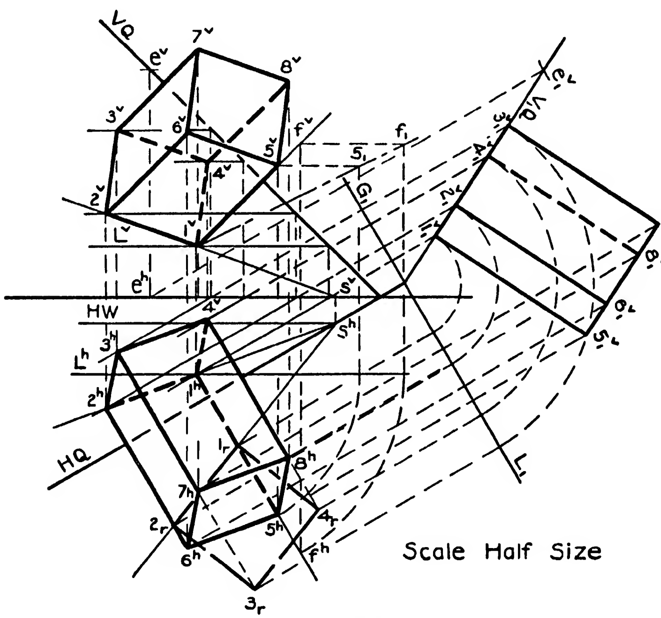

FIG. 254.

An alternative method for finding the long edges of the prism is also given in Fig. 254. From point 1 draw the line ($1^h f^h$, $1^v f^v$) perpendicular to Q and of unknown length. Find the true length $1^v f_1$. Make $1^v 5_1$ the required length, $1\frac{3}{4}''$; from 5_1 find the projections 5^v and 5^h (Prob. 30, First Analysis, § 148). The remaining long edges, 2–6, 3–7, and 4–8, may now be drawn equal and parallel to 1–5.

CHAPTER XVIII

TANGENT LINES AND PLANES—GENERAL PRINCIPLES

150. Curves. A **curve** may be defined as a line, no portion of which is straight.

A plane curve is one which lies wholly in a plane. If no part of the curve is plane, the curve is a **space curve**, or **curve of three dimensions.** Familiar examples of plane curves are circles and ellipses. An example of a space curve is the helix, as shown by a screw thread or a spiral spring. Space curves result usually, though not necessarily, when two curved surfaces of any kind intersect each other.

151. Projections of Curves. The projection of a plane curve on any plane of projection is, in general, a curve. If, however, the plane of the curve is perpendicular to the plane of projection, the projection of the curve is a straight line. The projection of a space curve on any plane is always a curve.

Conversely, if one projection of a curve is a straight line, the curve is a plane curve. But if both projections of a curve are curves, the actual curve may or may not be a plane curve, so that no general rule can be given.

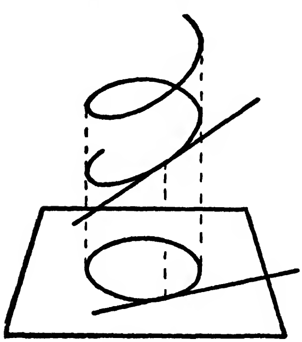

FIG. 255.

152. Tangent Lines. At any point in a curve, a straight line tangent to the curve can be drawn. The projection on any plane of the curve and its tangent are tangent to each other. (See Fig. 255.) An exception occurs if the plane of projection is perpendicular to the rectilinear tangent, since the latter then projects as a point.

153. Tangents to Plane Curves. A straight line tangent to a plane curve lies in the plane of the curve.

Hence, if it is required to draw a straight line tangent to a plane curve, from a point not on the curve, there is no solution unless the given point lies in the plane of the curve. In other words, a straight line tangent to a plane curve cannot be drawn from a point in space which lies outside the plane of the curve.

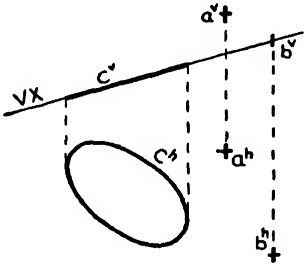

FIG. 256.

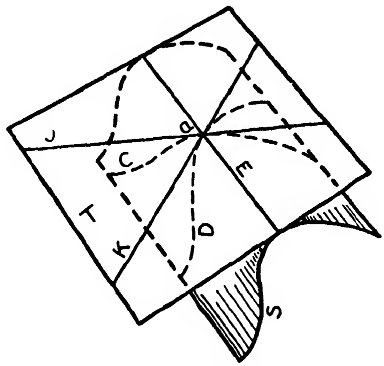

FIG. 257.

For example, Fig. 256 represents the projections of a plane curve C (§ 151), and of two points a and b. A line tangent to the curve C cannot be drawn from the point a, since it is obvious that a is not in the plane of the curve. Why? On the other hand, the point b lies in the plane of the curve, and from it two tangents (not shown in the figure) may be drawn to the curve C.

154. Tangent Planes to Curved Surfaces. Let S, Fig. 257, represent any curved surface. Let a be any point on the surface, and let T be the tangent plane to the surface at the point a.

Let C be any curve drawn on the surface through the point a. If drawn at random, C will probably be a space curve, although it may be taken so as to be a plane curve. Let J be the line tangent to the curve C at the point a. Then J will be tangent to the surface S, and will lie in the tangent plane T. Let D be a second curve lying on S and passing through

a. Then a line *K*, tangent to *D* at *a*, will also lie in the plane *T*.

It follows that the **tangent plane** to a surface at a point on it may be defined as the plane which contains all the straight lines tangent to the given surface at the given point of the surface.

155. Determination of Tangent Planes by Means of Tangent Lines. Two intersecting straight lines determine a plane. Suppose the given surface to be such that two curves of known properties can be drawn on the surface through the required point of tangency. The tangent plane is then determined by finding the plane which contains the two rectilinear tangents to these curves. In general, the curves used will be plane curves.

If the surface is such that a straight line, as *E*, Fig. 257, can be drawn on the surface through the given point *a*, the line *E* will lie in the tangent plane *T*. Such a straight line which lies wholly on the surface is called a **rectilinear element** of the surface. In this case, but one tangent line in addition to the element *E* is necessary to determine the plane *T*.

Curved surfaces exist in which *two* rectilinear elements of the surface can be drawn through a given point in the surface. For such surfaces the tangent plane at any point is the plane containing the two rectilinear elements passing through the point.

156. The Normal. The **normal** to a surface at any point on it is the straight line perpendicular to the tangent plane at that point. For certain curved surfaces, for example, the sphere shown in Fig. 258, the normal can be determined readily from known properties of the surface. Hence the tangent plane *T* at the point *a* can be found by drawing first the normal *N*, and then passing through the point *a* the plane *T* perpendicular to the line *N*.

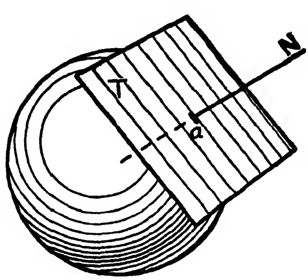

FIG. 258.

157. Determination of Tangent Planes by Means of the Normal. It is to be noted that the tangent plane T is absolutely determined in space by means of the normal N, since but one plane can be passed through a given point a perpendicular to a given line N. But in a projection drawing, where the plane T is not determined until its traces HT and VT are found, additional lines are necessary. This is shown in Fig. 259. The point a represents the given point in some given surface, and the line N represents the normal at that point. The tangent plane T is passed through the point a perpendicular to N. But although the projections N^h and N^v give the *directions* of HT and VT respectively (§ 112), the normal alone furnishes no point on either of these traces. Hence at least one line, as J, lying *in* the tangent plane, is required in the projection drawing to supply a necessary point, as s. (See Prob. 10, § 115.)

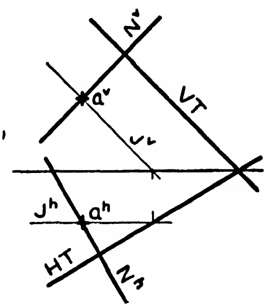

FIG. 259.

In determining the traces of a tangent plane by the use of the normal, therefore, the working method consists in finding, in addition to the normal, one line which is tangent to the given surface. The required tangent plane is then passed through the tangent line perpendicular to the normal.

CHAPTER XIX

TANGENT PLANES TO CONES AND CYLINDERS

158. The Cone and Cylinder. Definitions. The terms **cone** and **cylinder** are variously used, both in mathematical and in popular language, to denote either surfaces, or solids bounded in part by these surfaces.

As a *surface*, a **cone** may be defined as generated by a moving straight line, indefinite in extent, which always passes through a fixed point in space, and is so guided, by a curved line or otherwise, that a plane is not formed. A **cylinder** differs from a cone in that the generating straight line, instead of passing through a fixed point, remains always parallel to its initial position. In both of these surfaces, any position of the generating straight line is known as a **rectilinear element,** or simply as an **element,** of the surface.

The solids understood by the terms cone and cylinder are familiar objects, and do not need to be defined here. It should be remembered, however, that the axis of a cone or cylinder is not necessarily at right angles to the base.

159. Representation of the Cone and the Cylinder. We shall consider at present only the solid forms of these objects, and shall confine ourselves to those in which the base is either a circle or an ellipse. The projections (C^h, C^v) of a general cone, in which the base is an ellipse, and the axis oblique to the plane of the base, are given in Fig. 260.

We shall not attempt to solve problems, however, when the cone is projected as in Fig. 260. Since the base of the cone is a plane curve, it can be projected as a straight line (§ 151). In Fig. 260, let B (HB, VB) be the plane of the base, found by passing a plane through any three points of the curve (Prob. 6, Cor. II, § 106; construction not shown). Assume a second-

ary ground line perpendicular to HB; then the plane B will project edgewise as V_1B (see Fig. 92, § 70). Consequently, in

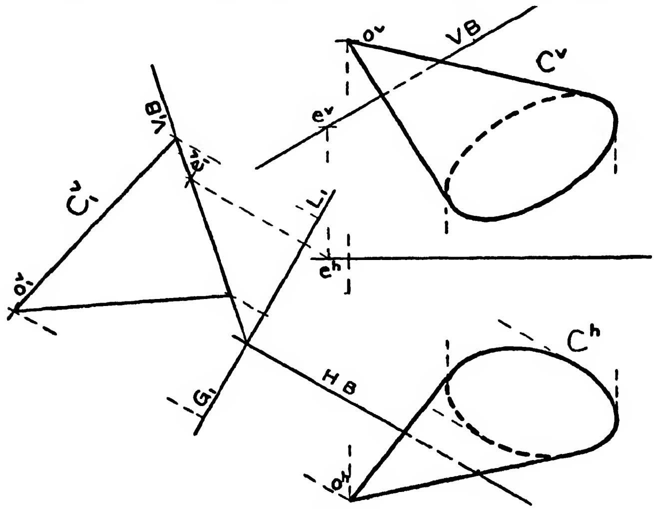

FIG. 260.

the secondary projection, $C_1{}^v$, of the cone, the base will project as a straight line.

Since this transformation always can be effected, we shall discuss further only those cases of cones and cylinders in which one projection, at least, of the base is a straight line. This results when the plane of the base is perpendicular to, or coincident with, one of the coördinate planes.

160. Projections of Cones and Cylinders. We shall place these objects in the first or third quadrants only (see § 19). Projections of cones are given in Figs. 261–265. These figures represent the following objects:

Fig. 261. *Cone in the third quadrant; base in a plane perpendicular to V; base visible.* Note that if we attempt to read these as the projections of a cone in the first quadrant, the base should be invisible. As this is not the case, the cone cannot be in the first quadrant. In this and similar cases, the student will often be expected to determine the quadrant from the given visibility of the projection.

Fig. 262. *Cone in the third quadrant; base in a plane perpendicular to H; base invisible.*

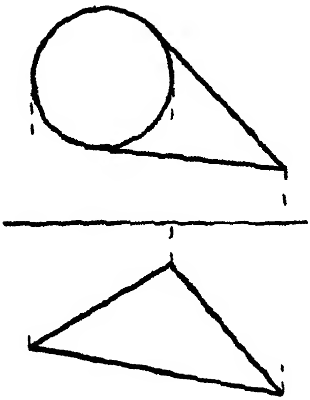

Fig. 261.

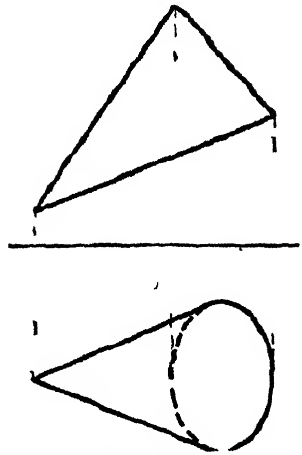

Fig. 262.

Fig. 263. *Cone in the first quadrant; base in H; base invisible.*

Fig. 264. *Cone in the third quadrant; base in V; base visible.*

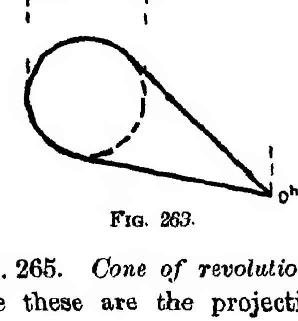

Fig. 263.

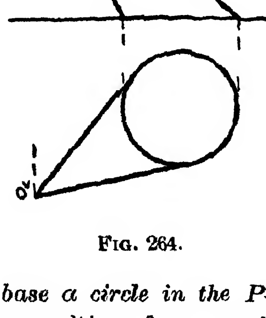

Fig. 264.

Fig. 265. *Cone of revolution; base a circle in the P-plane.* While these are the projections resulting from projecting

this object, so placed, they are, by themselves, and without further information, ambiguous. They should be supplemented by the profile projection, and the quadrant in which the object lies must be indicated in some manner, as for example, by lettering the vertex.

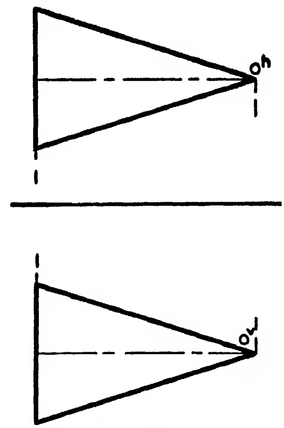

FIG. 265.

In projecting a cylinder, only one base will, in general, be shown, the cylinder being left indefinite in extent in one direction. Cylinders are projected in Figs. 266–270, which represent the following objects:

Fig. 266. *Cylinder in the third quadrant; base in a plane perpendicular to V; base visible.* As in the corresponding case of the cone (Fig. 261), an attempt to read these views as the projections of a cylinder in the first quadrant fails.

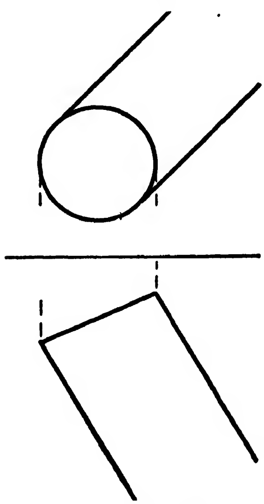

FIG. 266.

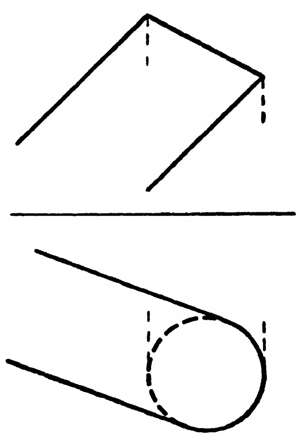

FIG. 267.

Fig. 267. *Cylinder in the third quadrant; base in a plane perpendicular to H; base invisible.*

Fig. 268. *Cylinder in the first quadrant; base in V; base invisible.*

Fig. 269. *Cylinder in the third quadrant; base in H; base visible.* The elements of this cylinder are all parallel to *V*.

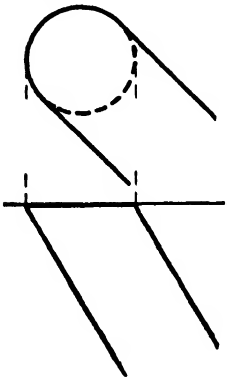

FIG. 268.

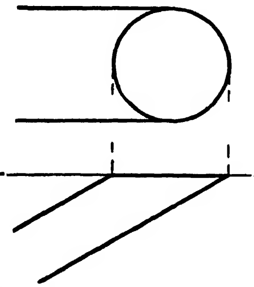

FIG. 269.

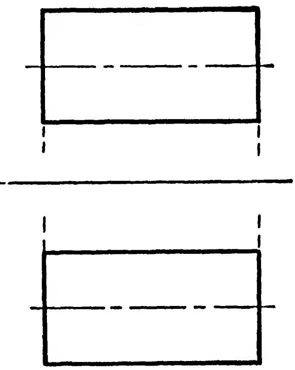

FIG. 270.

Fig. 270. *Cylinder of revolution; base a circle lying in P.* Like the cone of revolution (Fig. 265), these projections are ambiguous, and similar remarks apply.

161. Projection of a Point in the Surface of a Cone or Cylinder. Only one projection of a point lying in a conical or cylindrical surface can be assumed. The other projection of the point may be found by means of an element of the surface, as follows.

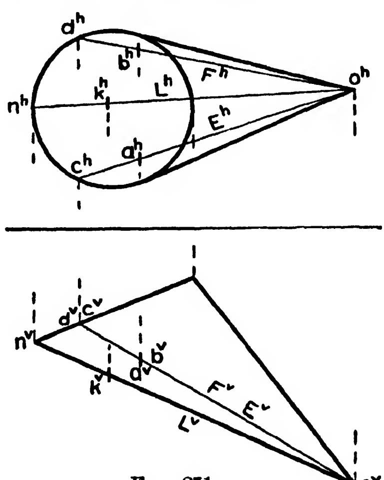

FIG. 271.

Let a cone be given as in Fig. 271, and let a^v be an assumed projection. The element *E*, passing through *a*, must also pass through the vertex *o*. Hence draw E^v through a^v and o^v. Produce E^v to meet the *V*-projection of the base

at c^v. Project from c^v to c^h in the H-projection of the base. Then E^h is drawn by connecting c^h and o^h. Find a^h in E^h by projecting from a^v. A second result is possible, namely, point b lying in the element F.

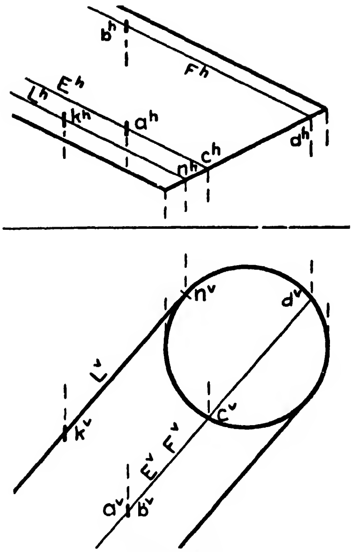

FIG. 272.

In general, an assumed projection will represent two points. But let k^v be assumed in the outside or contour element L. Then k^v is the projection of but one point in the surface.

A similar construction for the cylinder is shown in Fig. 272, the assumed projections being the double point a^vb^v, and the single point k^v.

In Fig. 273, the assumed projections are a^h, b^h. The base of the cone lies in P, which necessitates the use of the P-projection of the base. (Compare Fig. 265.) A similar construction applies to a cylinder whose base lies in P.

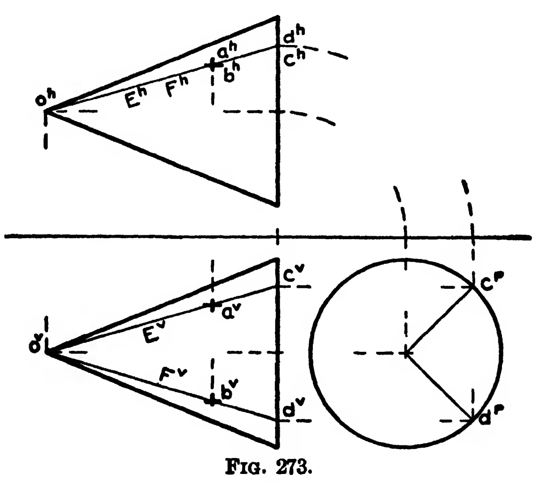

FIG. 273.

162. Tangent Planes to Cones and Cylinders. At every point in the surface of a cone or cylinder, a plane tangent to the surface can be drawn.

Planes tangent to cones and cylinders can also be passed to fulfill certain other conditions. Thus, the tangent planes may be required to contain a given point outside the surface, or to be parallel to a given straight line.

In determining these tangent planes, we shall make use of the following propositions.

(*a*) Through every point in the surface of a cone or cylinder, a rectilinear element can be drawn, and this element will lie in the tangent plane at that point (§ 155).

(*b*) It is a property of both the cone and cylinder that a plane tangent at any point in a given element is tangent at every point in the element. Hence, if the element of tangency is known, a second line in the tangent plane may be drawn tangent to the cone or cylinder at any point in this element. A convenient point is generally the point in which the element of tangency intersects the base.

(*c*) If the base of a cone or cylinder lies in *H*, any line tangent to the base lies in *H* (§ 153), and thus becomes the *H*-trace of a plane tangent to the surface. Similarly if the base lies in *V* or *P*.

(*d*) Every plane tangent to a cone contains the vertex.

Problem 31. ***To pass a plane tangent to a cone at a given point in the surface.***

Analysis. The plane is determined by the element which passes through the given point, and a line tangent to the base at the point where this element intersects the base (§§ 155, 162, *b*).

Construction. Case I. *The base of the cone lies in H or V.*

Example 1 (Fig. 274). The base of this cone lies in *H*. Let a (a^h, a^v), lying in the element E (E^h, E^v) be the given point in the surface. Since the required tangent plane contains the element E, find the traces s_1 and t_1 of E. Then the *H*-trace, HQ, of the tangent plane passes through s_1, and is determined

by the fact that HQ must be tangent to the base of the cone (§ 162, *c*). The trace VQ is now determined, since it must pass through t_1 and the point in which HQ intersects GL.

Let it also be required to find the tangent plane at the point b, lying in the element F. The H-trace, HT, of this plane,

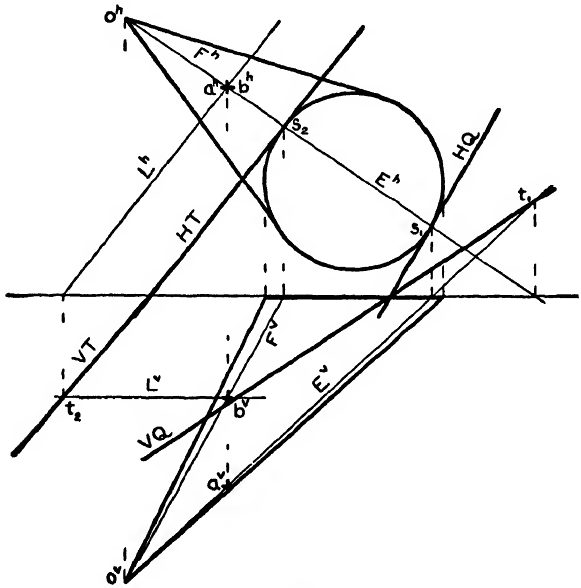

FIG. 274.

passes through the H-trace, s_2, of F, and is tangent to the base of the cone. The V-trace, VT, passes through the V-trace of F, but this point is inaccessible. Recourse must be had to some auxiliary line lying in the plane T, and so situated that its V-trace is accessible. (See § 108.) As HT is known, a horizontal principal line of the plane may be drawn through any point of the line F (§ 108, Ex. 3). A convenient point is the given point b. Through b^h draw L^h parallel to HT; through b^v draw L^v parallel to GL. Then the line L lies in the plane T, and the V-trace, t_2, of L is a point in VT.

EXAMPLE 2 (Fig. 275). The base of this cone lies in V. Tangent planes are passed at the points a and b. For the

plane Q, tangent at a, the trace VQ is drawn tangent to the base of the cone through the V-trace of the element E. Since the H-trace of the element E is not available, a point on HQ is found by means of the auxiliary line L, a vertical principal

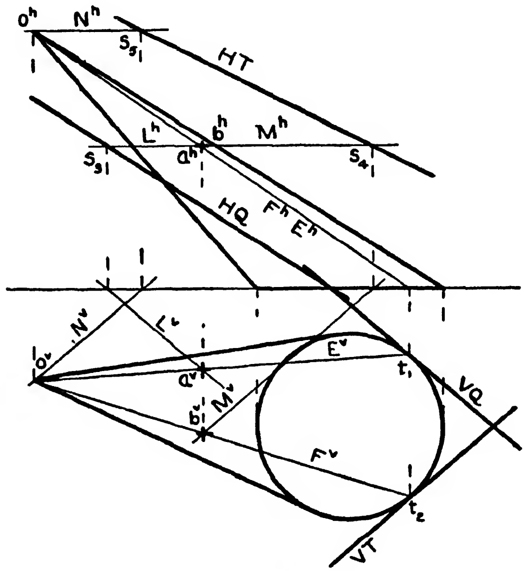

FIG. 275.

line of Q, drawn through the given point a. The H-trace of L lies in HQ.

The plane T is tangent at the point b. The trace VT is drawn tangent to the base of the cone, through the V-trace of the element F passing through b. In this case VT does not intersect GL within reach, neither is the H-trace of F obtainable. Hence two points on HT must be obtained by the use of auxiliary lines. Two convenient lines are the vertical principal lines, M and N, of T, drawn, one through the given point b, the other through the vertex, o, of the cone, since the vertex is in every tangent plane (§ 162, d). The H-traces, s_4 and s_5, of these two lines determine HT.

Case II. *The base of the cone lies in P* (Fig. 276). The base of this cone is a circle, and the cone is the symmetric form known as the cone of revolution. Let a (a^h, a^v) be the given point. Point a lies in the element E, which intersects the plane of the base in the point c, the profile trace of the line.

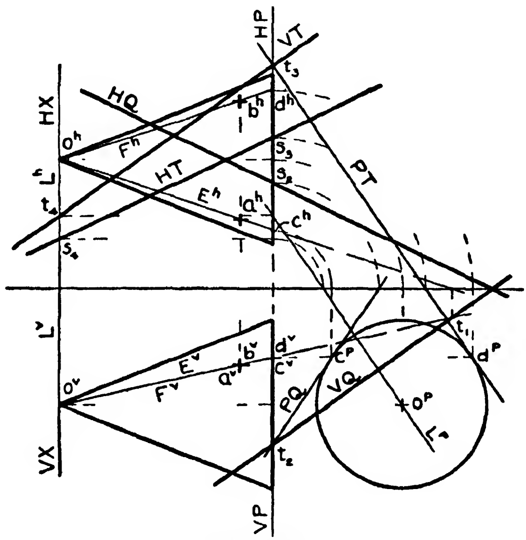

Fig. 276.

Through the actual trace c^p draw the profile trace, PQ, of the tangent plane, tangent to the profile projection of the base. From PQ are found the points, t_2 on VQ, and s_2 on HQ (§ 60). Since E lies in Q, find also, if possible, the H- and V-traces of E. In this case the V-trace, t_1, may be found, and this is sufficient; for VQ passes through t_2 and t_1, and HQ through s_2 and the point in which VQ intersects GL.

The plane T is tangent at the point b, lying in the element F. This element intersects the base at d. The profile trace, PT, is drawn through d^p tangent to the base. From PT are obtained the points s_3 on HT and t_3 on VT. Neither the H- nor V-trace of F is accessible, so that an auxiliary line is necessary. Since the vertex of the cone, o, is in every tangent plane (§ 162, d),

let an auxiliary profile plane, X, be passed through o. Since X is parallel to P, it will intersect the plane T in a line, L, parallel to PT. Project o to the plane P, and find the profile projection, o^p. Through o^p draw the P-projection, L^p, of the line L, parallel to PT. Find the traces of L; first on P, then by projection to X, giving the H-trace, s_4, and the V-trace, t_4. The traces HT and VT are now determined, since we have two points on each.

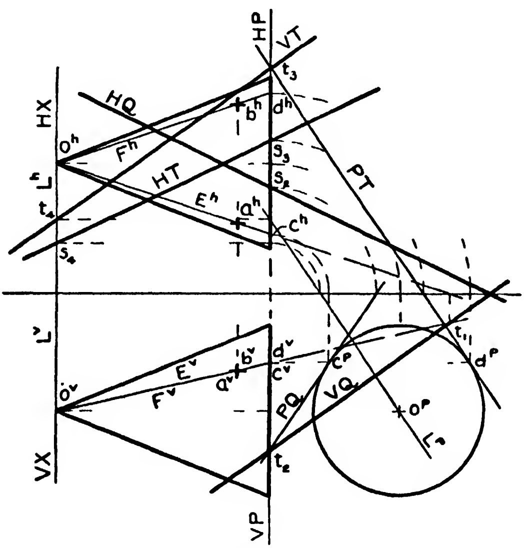

FIG. 276 (repeated).

CASE III. *The base of the cone does not lie in H, V, or P.* In this, the general case of this and the succeeding problems, one projection of the base will always be taken as a straight line. (See § 159.) The base of the given cone, Fig. 277, lies in a plane perpendicular to H and oblique to V. Let a be the given point of tangency. This point lies in the element E, which intersects the base in the point c. The plane tangent at a is also tangent at c; hence through c draw the line J, tangent to the base of the cone (§ 162, b). The required tangent plane, Q, is now determined by the lines E and J. In this case a sufficient number of the traces of E and J may be found to determine the traces HQ and VQ (Prob. 6, § 106).

The plane tangent at the point b will contain the element F passing through b. Line F intersects the base at d. Through d draw the line K tangent to the base; then K is a second line

in the tangent plane at b (§ 162, b). But in attempting to pass a plane through F and K, only the traces, s_3 and t_3, of K can be found; neither trace of F is available. Therefore, choose any point in F; a convenient point is the vertex, o.

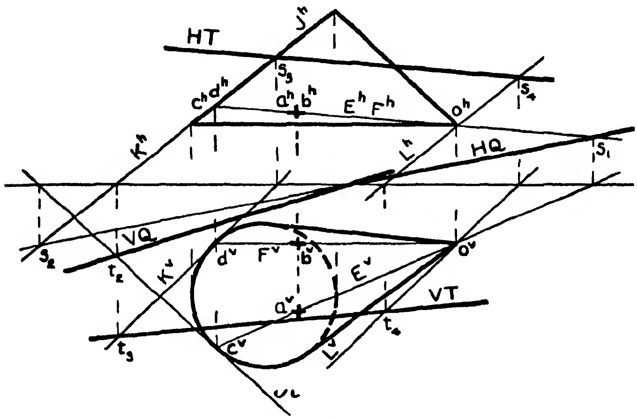

FIG. 277.

Through o draw the auxiliary line L, parallel to K; then L lies in the plane of F and K (§ 108, Ex. 1). Line L supplies the traces s_4 and t_4. These, taken in connection with the traces of K, are sufficient to locate the traces, HT and VT, of the required tangent plane.

Problem 32. ***To pass a plane tangent to a cone through a given point without the surface.*** (***Two results.***)

Analysis. Pass a line through the given point and the vertex of the cone. Find the point in which this line pierces the plane of the base of the cone. From this piercing point, draw a line tangent to the base of the cone. The required tangent plane is determined by the tangent line and the line first drawn. In general, two tangents may be drawn to the base, giving two possible tangent planes.

To prove this analysis, note that the vertex of the cone lies in every tangent plane (§ 162, d). Hence the line first drawn must lie in the required tangent plane. A tangent to the base

which intersects this line must also lie in the required tangent plane. But a line tangent to the base must lie in the plane of the base, hence it can be drawn only from that point of the first line in which it intersects the plane of the base (§ 153).

If the base of the cone lies in *H*, the point in which the line connecting the given point with the vertex pierces the plane of the base becomes the *H*-trace of this line, while the tangent

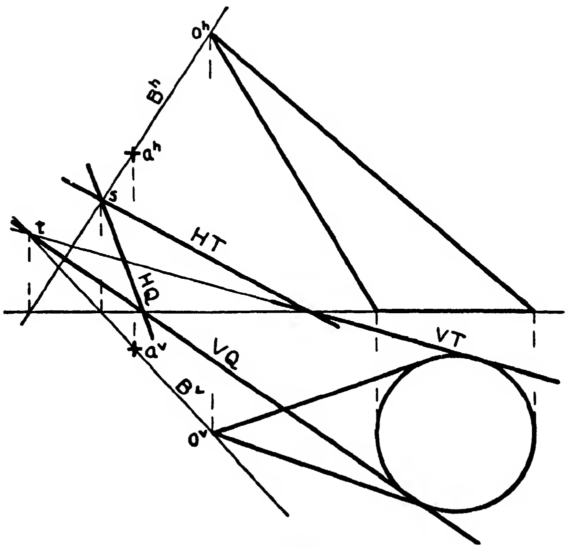

FIG. 278.

drawn from this trace becomes the *H*-trace of the tangent plane (§ 162, *c*). The construction is similar if the base lies in *V*.

Construction. CASE I. *The base of the cone lies in H or V* (Fig. 278). The base of the cone here given lies in *V*, and *a* (a^h, a^v) is the given point. Draw the line *B* (B^h, B^v), connecting *a* and the vertex *o*. Find the traces, *s* and *t*, of *B*. Since the base of the cone lies in *V*, draw *VQ* tangent to the base from the *V*-trace, *t*, of *B*. The trace *HQ* passes through the *H*-trace, *s*, of *B* (§ 98). A second tangent plane *T* (*HT*, *VT*) is similarly obtained.

CASE II. *The base of the cone does not lie in H or V* (Fig. 279). The base of the cone here given lies in the plane X, perpendicular to V. The given point is a. Draw the line B, connecting a with the vertex of the cone. Find the point, c, in which B pierces X. (See Fig. 178, § 119.) From c draw

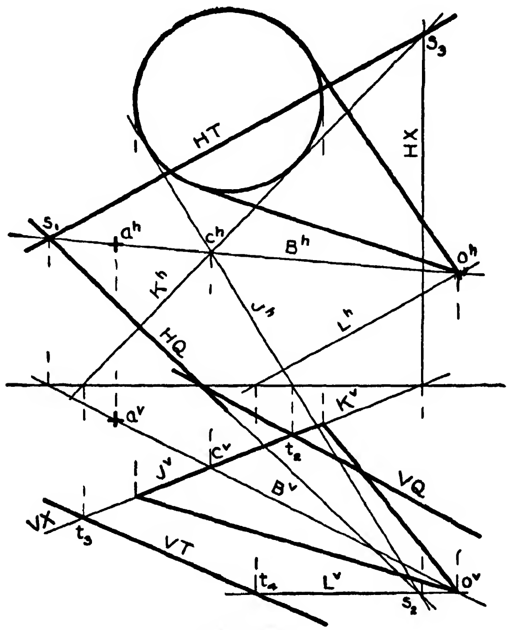

FIG. 279.

the line J (J^h, J^v) tangent to the base of the cone. The required tangent plane, Q, may now be passed through the lines B and J (Prob. 6, § 106). For this plane no auxiliary lines are needed. Draw also from c the tangent line K (K^h, K^v). Then a second tangent plane is determined by the lines B and K. For this plane, the H-trace, HT, is determined by the traces, s_1 and s_3, of B and K, respectively. One point of VT is t_3, the V-trace of K. A second point is t_4, which is found by using the auxiliary line L, drawn through the vertex of the cone parallel to HT (§ 108, Ex. 3).

Problem 33. ***To pass a plane tangent to a cone parallel to a given line.*** ***(Two results.)***

Analysis. The required tangent plane must contain the vertex of the cone (§ 162, *d*). Moreover, since it is to be parallel to a given line, the plane must contain a line parallel to the given line (§ 105). Hence, through the vertex of the cone

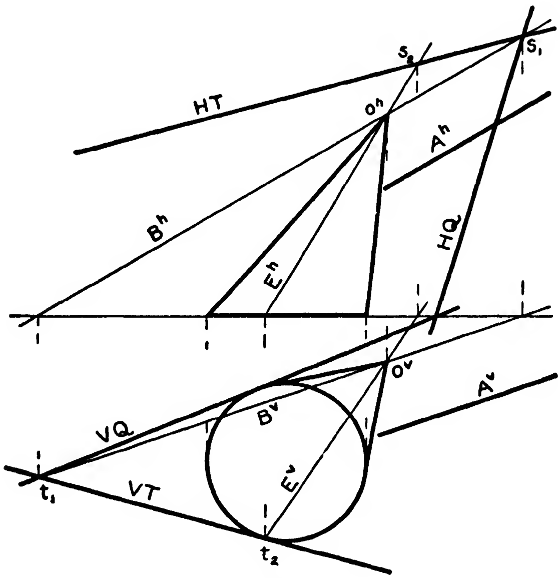

Fig. 280.

pass a line parallel to the given line. Pass the required tangent planes through this line. For the method of accomplishing this, see the preceding Problem.

Construction. Case I. *The base of the cone lies in H or V* (Fig. 280). The base of the given cone lies in *V*. The given line is *A*. Through the vertex, *o*, of the cone, draw $B\,(B^h, B^v)$ parallel to *A*. The *V*-trace of *B* is t_1. From t_1 draw *VQ* tangent to the base of the cone. The trace *HQ* passes through the *H*-trace, s_1, of *B*. For a second result, from t_1 draw *VT* tangent to the base of the cone. Since *VT* does not intersect

GL within reach, an auxiliary line is needed to locate *HT*. But *VT* is so nearly parallel to *GL* that the use of a vertical principal line of *T* will not give an especially accurate construction. Let us, then, note the point of tangency, t_2, of *VT* and the base. Through t_2 draw the element $E(E^v, E^h)$, the

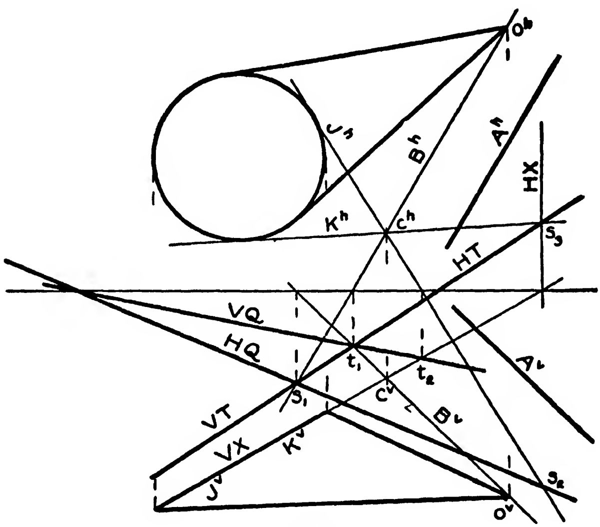

FIG. 281

element of tangency, which must lie in the plane *T*. Hence the *H*-trace, s_2, of *E* lies in *HT*, which is now determined by s_2 and the *H*-trace, s_1, of *B*.

CASE II. *The base of the cone does not lie in H or V* (Fig. 281). The base of the given cone lies in the plane *X*, perpendicular to *V*. The given line is *A*. Through the vertex, *o*, of the cone, draw *B* parallel to *A*. Line *B* intersects the plane *X* in the point *c*. From *c* draw the tangents, *J* and *K*, to the base of the cone. Plane *Q* is passed through *B* and the tangent *J*, while plane *T* is passed through *B* and the tangent *K*. No auxiliary lines are necessary. The planes *Q* and *T* are the required tangent planes.

NOTE. It will be seen in Figs. 280 and 281 that if the line B, parallel to A, intersects the plane of the bases of the cone within the circumference of the base, no tangent lines will be possible. Consequently, there will be no solution. This will happen when the line A is parallel, or nearly so, to the axis of the cone.

A special case, in which one, and only one, solution exists, will occur when the line B, parallel to A, is found to coincide with an element of the given cone.

Problem 34. ***To pass a plane tangent to a cylinder at a given point in the surface.***

Analysis. As in the corresponding problem with the cone (Prob. 31), the tangent plane is determined by the element

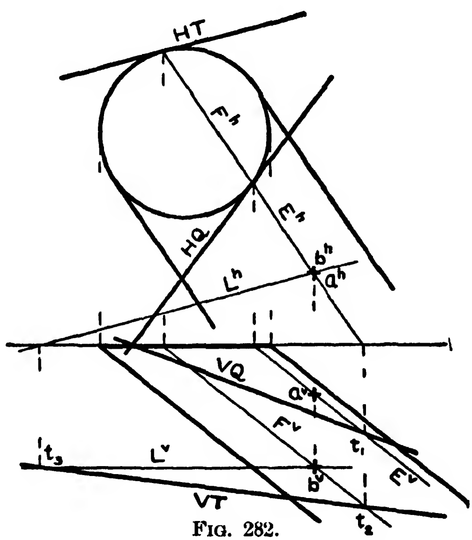

FIG. 282.

which passes through the given point, and a line tangent to the base at the point where this element intersects the base.

Construction. CASE I. *The base of the cylinder lies in H or V* (Fig. 282). The construction is entirely similar to that for the corresponding case of the cone (Prob. 31, Case I), and it should not require an extended explanation.

The plane Q, tangent at the point a, contains the element E. No auxiliary line is needed.

The plane T is tangent at the point b, lying in the element F. A necessary point, t_3, in VT, is here determined by the auxiliary line L.

Case II. *The base of the cylinder lies in P* (Fig. 283). The given cylinder is a cylinder of revolution, with a circular base lying in P. Obtain, if not already given, the profile projection of the base. For the plane Q, tangent at the point a, obtain

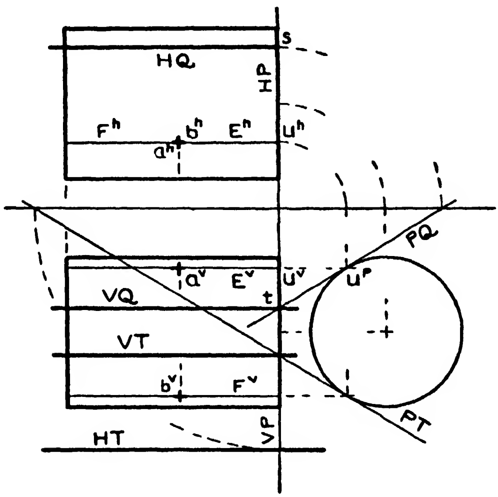

Fig. 283

the profile trace, u^p, of the element E containing a. Through u^p draw the profile trace, PQ, tangent to the base. This trace furnishes the point s on HQ and the point t on VQ. In this case no other points are necessary, since the plane Q is evidently parallel to GL. The plane T (PT, HT, VT), tangent at the point b in the element F, is obtained in a similar manner.

Case III. *The base of the cylinder does not lie in H, V, or P* (Fig. 284). The construction is entirely similar to that for the corresponding case of the cone (Prob. 31, Case III), and should be compared with it. A detailed explanation will not be given.

The required planes are Q, tangent at the point a, and T, tangent at the point b.

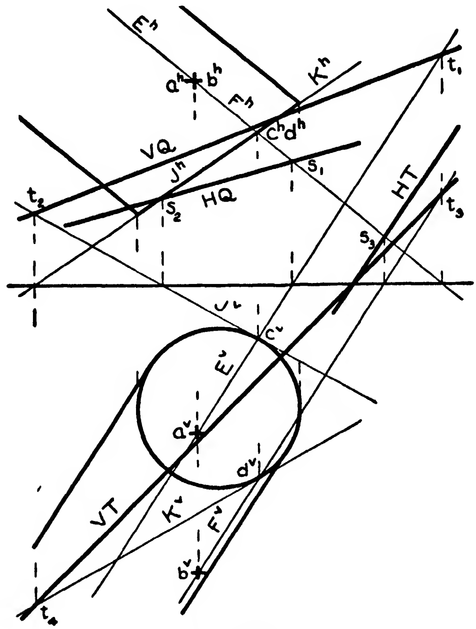

Fig. 284.

Problem 35. ***To pass a plane tangent to a cylinder through a given point without the surface.*** ***(Two results.)***

Analysis. Through the given point pass a line parallel to the elements of the cylinder. Since the required tangent plane contains some element of the surface, this line must lie in the required tangent plane. Find the point in which this line pierces the plane of the base of the cylinder. From this point draw a line tangent to the base. Pass the required tangent plane through the tangent line and the line first drawn. In general, two tangent lines, and hence two tangent planes, are possible.

Construction. Case I. *The base of the cylinder lies in H or V* (Fig. 285). The base of the given cylinder lies in H. The given point is a. Through a draw the line $B\,(B^h,\ B^v)$ parallel

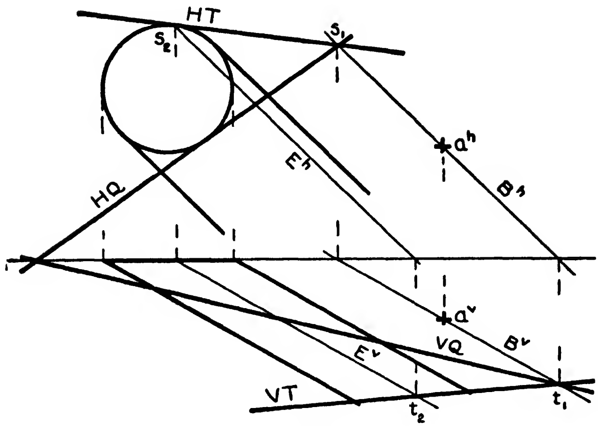

FIG. 285.

to the elements of the cylinder. From this on the construction is the same as for the corresponding case of the cone (Prob. 32, Case I). The required planes are Q and T.

CASE II. *The base of the cylinder does not lie in H or V* (Fig. 286). The base of the given cylinder lies in the plane Z, perpendicular to H. Through the given point a, draw the

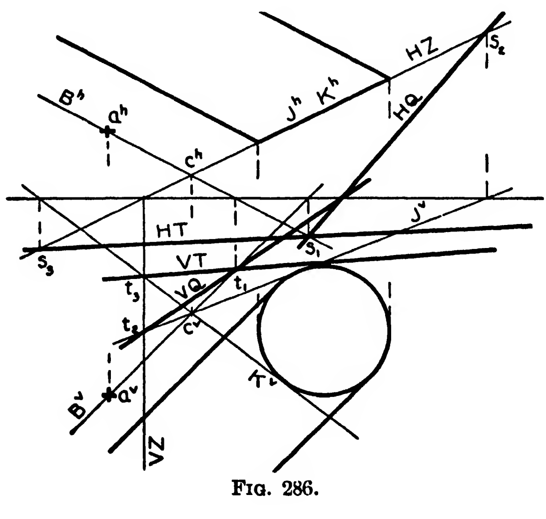

FIG. 286.

line B parallel to the elements of the cylinder. Then proceed as in the corresponding case of the cone (Prob. 32, Case II). The resulting planes are Q and T.

Problem 36. ***To pass a plane tangent to a cylinder parallel to a given line.*** *(Two results.)*

Analysis. No point of the required tangent plane is known in advance, as in the corresponding problem with the cone (Prob. 33); hence the result must be accomplished by indirect methods. Through the given line, pass a plane parallel to the elements of the cylinder. This is possible, since all the elements are parallel. This plane will then be parallel to

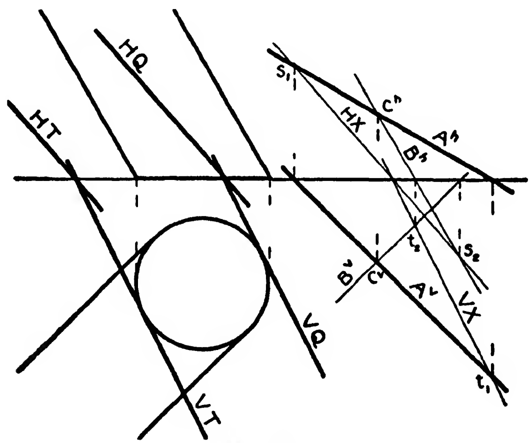

FIG. 287.

the required tangent planes, and may be moved, parallel to itself, until tangent to the cylinder. The manner of accomplishing this is best shown in the construction.

Construction. CASE I. *The base of the cylinder lies in H or V* (Fig. 287). The base of the given cylinder lies in V. The given line is A. Through A pass the plane X parallel to the elements of the cylinder. To do this, assume any point, c, in A; through c draw the line B, parallel to the cylinder; pass the plane X through the lines A and B. (See Prob. 7, § 107.) Since the base of the cylinder lies in V, VQ is now

drawn tangent to the base parallel to *VX*; *HQ* is then drawn parallel to *HX*. The traces *VT* and *HT* of the second required tangent plane are obtained in a similar manner.

CASE II. *The base of the cylinder lies in P* (Fig. 288). The base of the given cylinder is a circle whose center is *o*; find the *P*-projection of this circle. As in Case I, through the given line *A* pass the plane *X*, parallel to the elements of the cylinder. Since the base of the cylinder lies in *P*, find the profile trace, *PX*, of this plane. Draw *PQ* and *PT* parallel to *PX*, tangent to the base of the cylinder; these are the profile

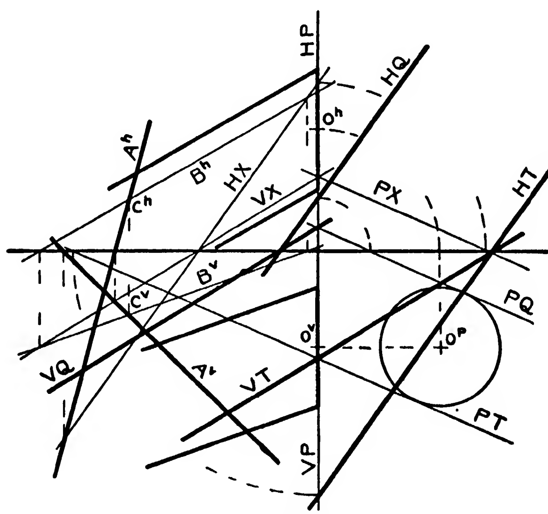

FIG. 288.

traces of the required tangent planes. From *PQ* and *PT* are obtained the *H*- and *V*-traces of the planes, *HQ* and *HT* being parallel to *HX*, while *VQ* and *VT* are parallel to *VX*.

CASE III. *The base of the cylinder does not lie in H, V, or P* (Fig. 289). The base of the given cylinder lies in the plane *Z*, perpendicular to *V*. The given line is *A*. Through this line is passed, as in the previous cases, the plane *X*, parallel to the elements of the cylinder. The next step is to draw a line, *J*, which shall be tangent to the base of the cylinder and at

the same time parallel to the plane *X*. This line must lie in the plane, *Z*, of the base of the cylinder (§ 153); it must also be parallel to some line in the plane *X* (§ 104); hence *J* will be parallel to the line of intersection of the planes *X* and *Z*. Find this line of intersection, *L* (Prob. 12, § 118). Draw the line *J* tangent to the base of the cylinder parallel to the line

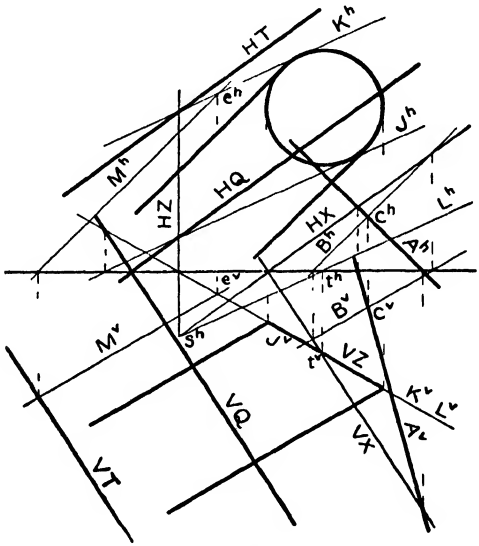

FIG. 289.

L. Through *J* pass the required tangent plane, *Q*, parallel to *X*.

A second result, plane *T*, is found by means of the tangent line *K*, also parallel to *L*. Here *VT* is not located either by the *V*-trace of *K* or the point where *HT* intersects *GL*. The construction used is to assume some point, as *e*, on *K*. Through *e* draw the line *M*, parallel to the elements of the cylinder. Then the line *M* lies in the plane *T*, and the *V*-trace *t* is a point on *VT*.

CHAPTER XX

TANGENT PLANES TO DOUBLE CURVED SURFACES OF REVOLUTION

163. Double Curved Surfaces of Revolution. A **double curved surface of revolution** is a surface formed by the revolution of any curve about any straight line as an axis, provided the resulting surface is such that no straight lines can be drawn on it.

Familiar solids whose surfaces are double curved surfaces of revolution are the sphere, the various ellipsoids, and the torus (§ 26).

164. Representation of Double Curved Surfaces of Revolution. In solving problems involving double curved surfaces of revolution, we shall place the axis of the surface perpendicular to one of the coördinate planes. If not so given, the transformation can be effected by new planes of projection.

In Fig. 290 is shown a general case, a vase whose outer surface is a double curved surface of revolution. The axis is placed perpendicular to *H*.

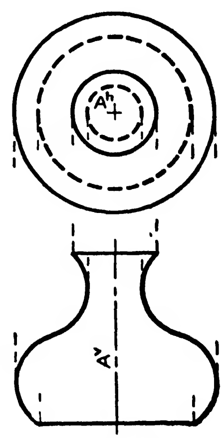

Fig. 290.

165. Meridians. The *V*-projection, Fig. 290, shows the true shape of the section made by a plane containing the axis of the surface. Such a section is called a **meridian section**, or simply a **meridian.** Any plane which contains the axis is called a **meridian plane.** All meridians are alike, and the surface is usually considered as formed by the revolution of its meridian section.

The meridian which forms the outline of a projection of the surface, in Fig. 290 the V-projection, is called the **principal meridian.** The plane which contains this meridian is called the **principal meridian plane.** The principal meridian plane is always parallel to one of the coördinate planes; in Fig. 290 it is parallel to V.

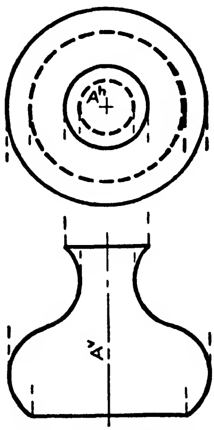

FIG 290 (repeated).

166. Parallels. Let a double curved surface be formed by the revolution of its meridian section about the axis. Each point of the generating meridian describes a circle lying in a plane perpendicular to the axis (§ 74). These circles are called **parallels** of the surface.

The projection of a double curved surface of revolution on a plane perpendicular to its axis will consist of one or more circles, which are the projections of particular parallels of the surface (§§ 75, 84). For example, see the H-projection of the vase, Fig. 290.

167. Projections of a Point in a Double Curved Surface of Revolution. Through every point of a double curved surface of revolution a circle which is a parallel of the surface (§ 166) can be drawn A point will lie in the surface if its projections lie in the corresponding projections of a parallel of the surface. With the axis of the surface perpendicular to either H or V, one projection of the parallel will be a circle, the other projection a straight line (§ 75).

For example, consider the ellipsoid shown in Fig. 291, where the axis of the ellipsoid is perpendicular to H. Let a^h be a given projection of a point in the surface. Through a^h draw the H-projection, a circle, R^h, of the parallel passing through a. Find the V-projection of the circle R, namely, the straight line R^v. Project from a^h to a^v in R^v; then point $a\,(a^h, a^v)$ lies in the surface. The given H-projection may also represent a point b, lying on the symmetric parallel S (S^h, S^v).

Let c^v be given. This point lies on the parallel T, whose V-projection, T^v, is a straight line through c^v. The H-projection of this parallel is the circle T^h. In T^h is found c^h by projection from c^v. There is also a second result; namely, the point d (d^v, d^h)

If a surface is wholly convex, as in this example, any point assumed in either projection represents two points in the surface, unless the assumed projection lies in the *contour* (outline) of the surface. Thus, if e^v is assumed in the V-projection, M^v, of the principal meridian (§ 165), the single H-projection, e^h, lies in the straight line M^h, which is the H-projection of the principal meridian. If f^h be assumed in the H-projection, P^h, of the greatest parallel (§ 166), f^v lies in the straight line P^v, the V-projection of this parallel.

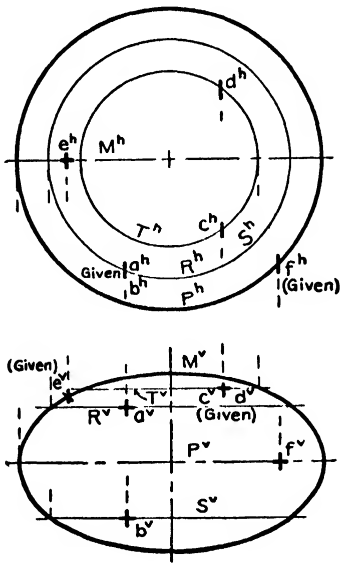

Fig. 291.

168. The Sphere. The sphere possesses the unique property that every plane section is a circle. All planes which contain

the center of the sphere intersect the surface in circles of the same size, known as great circles. Any diameter of the sphere may be taken as its axis. In the solution of problems involving the sphere, advantage is usually taken of some or all of these properties. The solution thus becomes a particular solution, and the sphere, in consequence, is not a good surface to use to illustrate the general case of a double curved surface of revolution.

In the problems of tangencies which follow, we shall treat the sphere as a particular case, apart from the general problem.

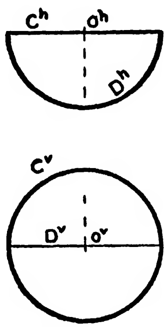

FIG. 292.

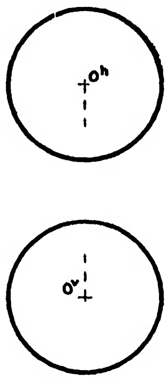

FIG. 293.

On account of the perfect symmetry of the sphere, and the possibility of taking any diameter as an axis, students often have more difficulty in visualizing the relation of points on the surface than with a less symmetric surface. This difficulty usually disappears by visualizing first a hemisphere, as shown in Fig. 292. This figure represents the front half of a sphere, bounded by the great circle C (C^v, C^h), and the portion of the surface represented by each projection appears more clearly than when the entire sphere is given, as in Fig. 293.

169. The Torus. The surface of a torus (Fig. 294) will usually be taken to illustrate the general case of a surface of revolution. This surface is divided by the circles C and D

into two portions: the so-called outer portion, doubly convex, where a tangent plane contains but a single point of the surface, and an inner portion, concavo-convex, where a tangent plane at any point also intersects the surface.

Certain properties of this surface have already been given. Thus in § 86, it has been shown that any plane perpendicular to the axis between C and D cuts the surface in two circles A

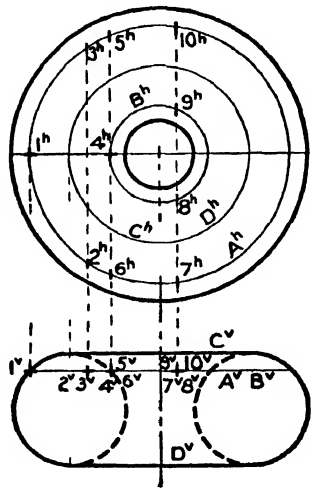

FIG. 294.

and B, which are parallels of the surface. Hence, if the surface is given as in Fig. 294, points assumed in the V-projection may represent, according to their position, one, two, three, or four points lying in the surface. A point assumed in the H-projection, however, as in the ellipsoid, Fig. 291, can never represent more than two points in the surface.

170. Tangent Planes to Double Curved Surfaces of Revolution. (*a*) A plane tangent to a double curved surface of revolution is tangent, in general, at but a single point. Exceptions occur, however; for example, in the torus, Fig. 294, the planes which contain the circles C and D are tangent at every point in these circles.

(*b*) Through every point in the surface two curves of known properties can be drawn, namely, the meridian (§ 165) and the parallel (§ 166) which pass through the given point.

(*c*) Planes which are tangent to points lying in the same parallel of the surface intersect the axis at the same point.

(*d*) The normal (§ 156) at any point of the surface lies in the meridian plane passing through the point.

(*e*) Every normal to the surface intersects the axis.

(*f*) Normals to points lying in the same parallel of the surface intersect the axis at the same point.

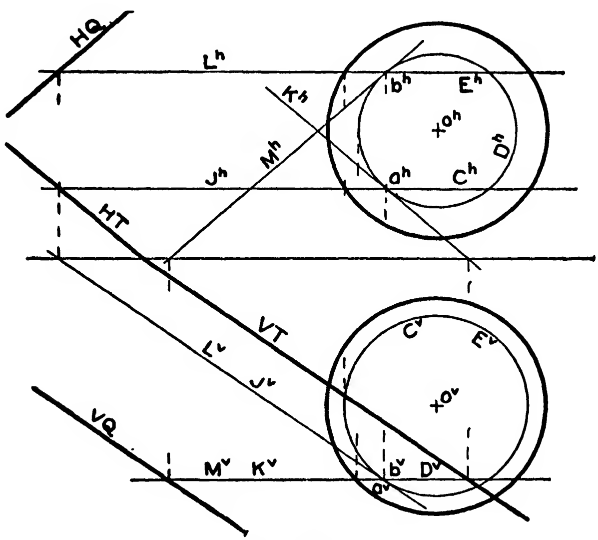

Fig. 295.

Problem 37. ***To pass a plane tangent to a sphere at a given point in the surface.***

Analysis. Through the given point, draw two circles lying in the surface. Draw tangent lines, one to each circle, at the given point. Find the plane determined by the two tangent lines. (See § 155.)

Construction (Fig. 295). Let a (a^h, a^v) be the given point. Through a we can draw two circles which have simple projec-

tions, namely the circle C (C^h, C^v) parallel to V and the circle D (D^h, D^v) parallel to H. Through a draw the line J (J^h, J^v), tangent to the circle C, and line K (K^h, K^v) tangent to the circle D (§ 152). The required tangent plane, T, is the plane determined by the lines J and K (§§ 102, 106).

A second result is the plane Q which is tangent at the point b (b^h, b^v) and obtained in a similar manner.

Problem 38. ***To pass a plane tangent to a double curved surface of revolution at a given point in the surface.***

First Analysis. The meridian and the parallel which pass through the given point are two known curves of the surface (§ 170, b). At the given point, draw tangent lines, one to each of these curves. Find the plane determined by the two tangent lines (§ 155).

Second Analysis. Draw the normal to the surface at the given point. Pass the required tangent plane through the point perpendicular to the normal (§ 156).

Third Analysis. Revolve the surface about its own axis until the given point is in the principal meridian plane. In this position the tangent plane will show edgewise as a line tangent to the outline of the surface at the given point. Counter-revolve the surface together with the tangent plane to the original position.

The actual construction, if made by the first analysis, must often be supplemented by the use of auxiliary lines to obtain points within the limits of the drawing. It has also been shown, in § 156, that while the normal alone determines the plane in space, auxiliary lines must be used to locate the traces in the drawing. But by the third analysis, it is possible in nearly every case to determine one trace of the tangent plane neatly and rapidly. We shall therefore begin the solution by the third analysis; then, to find the second trace of the plane, introduce as auxiliaries one or more of the lines indicated by the first and second analyses.

Construction (Fig. 296). The surface chosen to illustrate this, the general case, is that of a torus. Let a (a^h, a^v) be the

given point in the surface. Revolve the torus about its axis until a is in the principal meridian plane Y. The path of a will be the parallel C, and the resulting position a_r (a_r^h, a_r^v). At a_r^v we can draw by inspection the edge view VT_1 of the

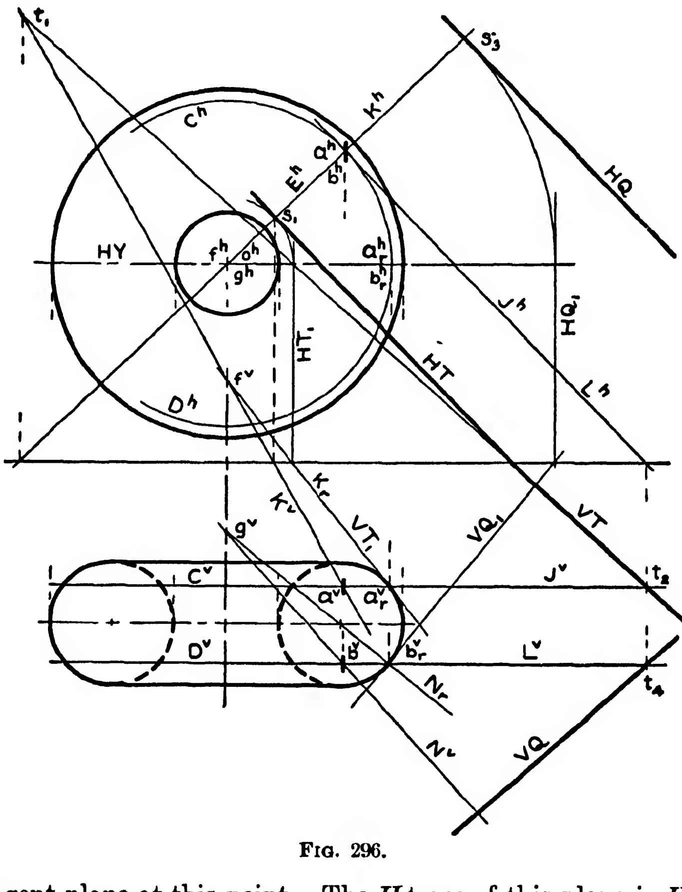

FIG. 296.

tangent plane at this point. The H-trace of this plane is HT_1. Counter-revolving the torus to the first position, HT_1 takes the final position HT (reverse of § 134). The tangent plane, as shown at VT_1, intersects the axis at the point f (f^v, f^h), which becomes a fixed point for all planes tangent at any point in the parallel C (§ 170, c). Hence, connecting f and a, we

have the line K (K^v, K^h), evidently tangent to the meridian E and lying in the tangent plane at a. The vertical trace, VT, of the required tangent plane is now determined by the V-trace, t_1 of K, together with the point in which HT intersects GL. In this case the line J, tangent to the parallel C at a, might be used instead of the line K.

Let b (b^h, b^v), Fig. 296, be a second point. Let b be revolved into the principal meridian at b_r (b_r^h, b_r^v). At b_r^v we may draw by inspection the edge view VQ_1 of the plane tangent at this point. This edge view does not intersect the axis within reach, as was the case with point a; but the plane Q_1 intersects H in the H-trace HQ_1. We can therefore obtain HQ by revolving HQ_1 about the axis of the torus, as was done for point a. The line L, tangent to D at b, is one line in the required tangent plane Q. This gives one point, t_4, in VQ. But since HQ does not intersect GL within the figure, VQ is still undetermined. Let us try the normal. Draw first the normal, N_r, at the point b_r^v. This normal intersects the axis at the point g^v. Then the normal at the point b passes through g (§ 170, f). Connecting g^v and b^v gives the V-projection, N^v. Since the plane Q is perpendicular to the normal N, VQ is now drawn through t_4 perpendicular to N^v (§ 112).

Where is the H-projection of the normal N? Note that this projection would not be needed in any event, since it would merely give the direction of HQ.

A second example, involving some additional features of construction, is given in Fig. 297. The given point is so chosen in the V-projection that it represents four points in the surface (§ 169). Two tangent planes are shown, one at an outside point, and one at an inside point. The plane Q is tangent at point a. This point lies in the parallel E (E^h, E^v), so that it revolves into the principal meridian plane at a_r (a_r^h, a_r^v). The trace HQ is obtained by revolving HQ_1 about the axis of the torus, as in Fig. 296. The meridian tangent K, and the line J tangent to the parallel E, give the V-traces t_1 and t_2, thus locating VQ. The normal, N, to the plane Q, although not needed, is shown in the V-projection; it here furnishes a

check on the construction, since N^v should be perpendicular to VQ (§ 112).

The plane T, Fig. 297, is tangent at the point c. This point lies in the parallel F (F^h, F^v); hence it revolves into the principal meridian at c_r (c_r^h, c_r^v). At c_r^v draw the edge view, VT_1, of the tangent plane, also the normal, U_r, to this plane. Then HT_1 revolves to HT. Since VT_1 intersects the axis at 3^v, the line M (M^v, M^h) is a line in the required tangent plane at c. But the line M gives no accessible point on VT; neither does the line L, drawn tangent to the circle F at point c. Draw the V-projection, U^v, of the normal to the plane T through 4^v and c^v (§ 170, f). Note that now but one point on VT is necessary. Why? Since the line M lies in the plane T, let the point 5 (5^h, 5^v) be some assumed point on M. Through the point 5 draw the line X (X^h, X^v) parallel to the line L (§ 108, Ex. 1). Find the V-trace, t_5, of X. Draw the required trace, VT, through t_5 perpendicular to U^v.

171. Tangent Planes which Contain a Given Line. A plane tangent to a double curved surface of revolution may, under certain conditions, be determined by the fact that the plane must contain a given straight line. For example, with surfaces like the sphere and the ellipsoids, it is evident that the necessary condition is that the line shall not intersect the surface; then, in general, two tangent planes are possible. But, whatever the form of the surface, if a solution exists in space, the traces of the plane can be found.

We are not prepared to take up at this time the general case of a plane through any line tangent to any double curved surface of revolution. The solution requires the use as an auxiliary of a surface whose properties have not yet been discussed. Particular solutions can be found, however, for the following cases.

(1) The given surface a sphere, the line general, in any position not intersecting the sphere.

(2) Any double curved surface of revolution, with the line in particular positions with respect to the axis of the surface.

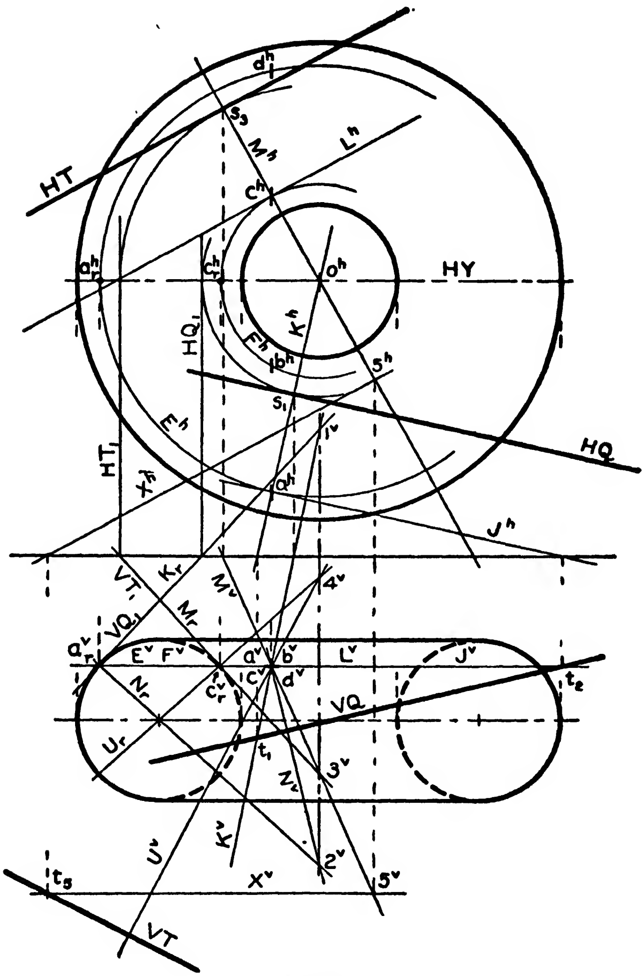

Fig. 297.

Problem 39. ***To pass a plane tangent to a sphere through a given line without the surface.***

Analysis. This analysis is shown pictorially in Fig. 298. Let A be the given line, and o the center of the given sphere. Let Q and T represent the required tangent planes. Pass an auxiliary plane, X, perpendicular to A, through the center of the sphere. This plane intersects A in the point c. Since X contains the center of the sphere, the intersection with the sphere is a great circle. This circle evidently contains the points of tangency of the planes Q and T. From c draw the line E tangent to the great circle. This can be done, since c and the circle are in the same plane, X (§ 153). Then E will lie wholly in the tangent plane Q, which is now determined, since we know two lines, A and E, lying in it. The plane T is similarly determined as the plane which contains the given line A and the tangent line F.

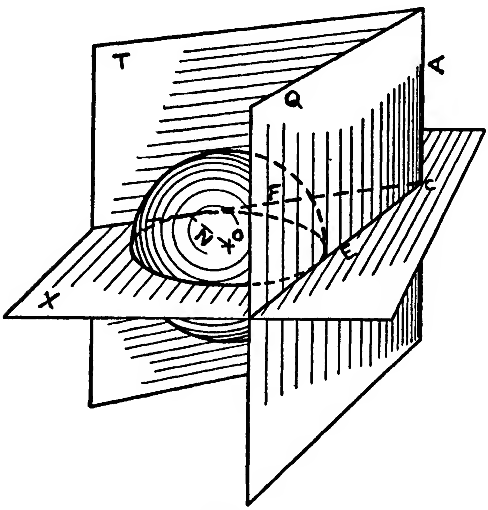

FIG. 298.

An alternative method for determining the plane T is also shown in Fig. 298. From o draw the line N perpendicular to F. This line lies in the plane X, and passes through the point of tangency of F and the sphere. The line N is thus the normal to the tangent plane T. Hence T may be determined as the plane which contains the given line A and is perpendicular to N. We may observe in passing that it is not possible, in general, to pass a plane through one line perpendicular to

another line, but the relative positions of the lines *A* and *N* are such in this case that it may be done.

Construction (Fig. 299). Let *A* be the given line, and *o* the center of the given sphere. Find the plane, *X*, which contains *o* and is perpendicular to *A* (Prob. 10, § 115). Find the point, *c*, in which *A* intersects *X* (Prob. 13, § 119). Since the

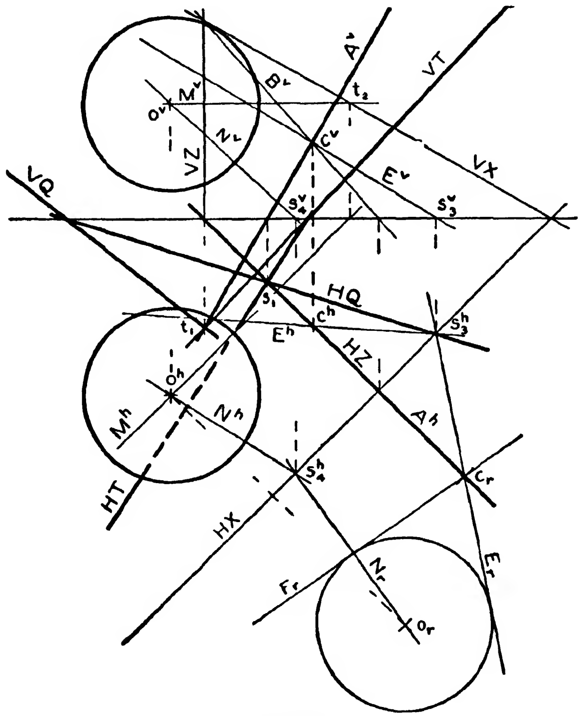

FIG. 299.

plane *X* is oblique to both *H* and *V*, the great circle in which *X* intersects the sphere will project as an ellipse in each view. To avoid drawing these ellipses, revolve the plane *X* about one of its traces into *H* or *V* (§ 138). Let *X* be revolved about *HX* into *H*. The point *c* revolves to c_r (Prob. 21, Working Rule, § 138); and the point *o* revolves to o_r. The great circle lying in *X* will now appear in true shape and size,

and may be drawn at once, with o_r as center. From c_r draw the two tangents, E_r and F_r, to this circle. These are two lines lying in the plane X (see the Analysis), and their projections may be obtained by counter-revolving the plane X (§ 147). The tangent E_r intersects HX at s_3^h, whence s_3^v is on

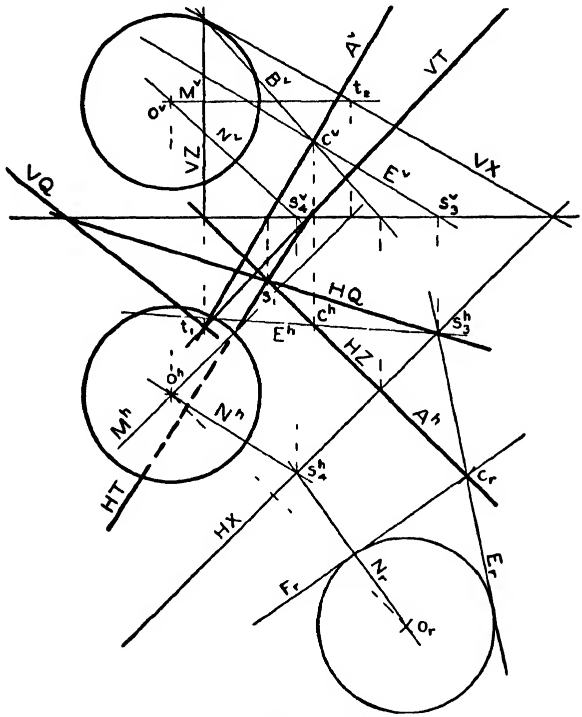

Fig. 299 (repeated).

GL, and the projections E^h, E^v are determined by the fact that the line E passes through the point c (reverse of Fig. 221, § 138). Pass the required tangent plane Q through the lines E and A (Prob. 6, § 106).

The tangent line F does not intersect HX within reach. Instead of attempting to find the projections of F, let us determine the tangent plane by means of the normal. Through o_r draw N_r perpendicular to F_r. The line N lies in the plane X

(see the Analysis); hence the intersection of N_r and HX is the H-trace of N. Using this trace and the point o, determine the projections, N^h and N^v, of N. The required tangent plane T contains A and is perpendicular to N. Let s_1 and t_1 be the traces of A. Through s_1 draw HT perpendicular to N^h; through t_1 draw VT perpendicular to N^v (§ 112).

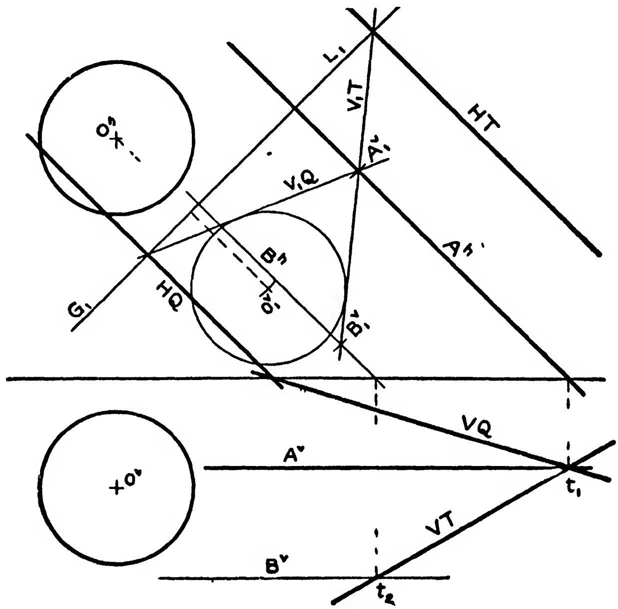

Fig. 300.

Special Case. *The given line is parallel to H or V.* Let the given line A, Fig. 300, be parallel to H. Assume a secondary ground line, G_1L_1, perpendicular to A^h. The line A will then project as a point, $A_1{}^v$ (§ 70). Project the center of the sphere to $o_1{}^v$, and draw the sphere. The edge views, V_1Q and V_1T, of the required tangent planes may now be drawn directly through $A_1{}^v$. The plane Q (HQ, VQ) requires no further explanation. The trace HT is readily obtained, but for VT an auxiliary line is necessary. The line used here is B, parallel to the line A. The projection $B_1{}^v$, a point, is assumed in V_1T. From $B_1{}^v$ are obtained B^h and B^v. The V-trace, t_2, of B locates a point in VT.

Problem 40. ***To pass a plane tangent to a double curved surface of revolution through a given line.*** **(*Special Cases.*)**

CASE I. *The given line intersects the axis of the surface.*

Analysis. Using the point in which the given line intersects

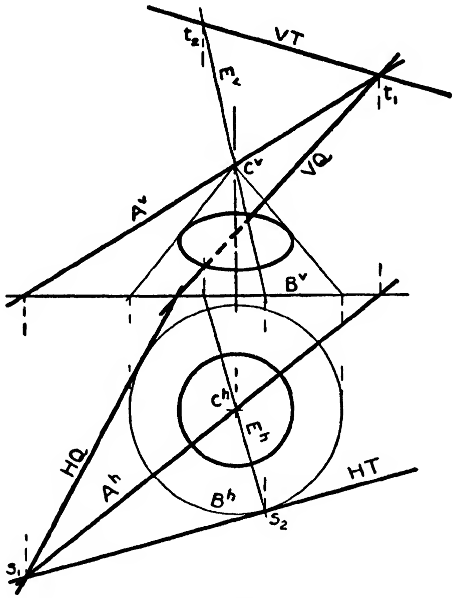

FIG. 301.

the axis as a vertex, circumscribe the given surface by a cone of revolution. Pass the required tangent planes through the given line tangent to the auxiliary cone.

Construction (Fig. 301). The given surface is an oblate spheroid, with its axis perpendicular to H. The given line A intersects the axis at the point c (c^h, c^v). From c^v draw tangents to the V-projection of the given surface; these tangents represent the auxiliary cone in the V-projection. The base of this cone on H is the circle B^h. The H-traces, HQ and HT, of the required tangent planes pass through the H-trace, s_1, of A, and are tangent to the circle B^h. (Compare Prob. 32, § 162.)

The V-traces, VQ and VT, pass through the V-trace, t_1, of A. A second point on VT is here obtained by finding the V-trace, t_2, of the element, E, in which the plane T is tangent to the auxiliary cone.

CASE II. *The given line is parallel to the axis of the surface.*

Analysis. The axis of the given surface being placed, as usual, perpendicular to one of the coördinate planes, for example H, the H-projection of the given line will be a point. The H-traces of the tangent planes will therefore be their edge views, and the planes may be drawn by inspection. We proceed similarly if the axis of the surface is perpendicular to V or P.

No figure to illustrate the construction is deemed necessary.

CASE III. *The given line is perpendicular to, but does not intersect, the axis of the given surface.*

First Analysis. Let the axis of the given surface be perpendicular to H. Then the given line is parallel to H. Revolve the line about the axis of the surface until the line is perpendicular to V, and projects as a point thereon. The edge views of tangent planes may now be drawn by inspection. Revolve these planes about the axis of the surface until they contain the original position of the given line. We proceed similarly if the axis of the surface is perpendicular to V or P.

Second Analysis. The given line may be projected as a point by choosing a secondary plane of projection perpendicular to the line. This plane necessarily will be parallel to the axis of the given surface. Hence the surface may be projected readily, and the edge views of the tangent planes may be drawn by inspection. The traces of the planes can then be obtained from the edge views.

Since the method of the second analysis necessitates the construction of an additional projection of the given surface, it is usually easier to employ the first analysis. The following construction is made by the first analysis.

Construction (Fig. 302). The given surface is that of a solid torus, only the outer (doubly-convex) portion of the curved

surface being retained. The axis of the surface is perpendicular to V. The given line A (A^v, A^h) is parallel to V. Revolve A about the axis of the torus to the position A_1 (A_1^v, A_1^h), where the H-projection, A_1^h, is a point. Through A_1

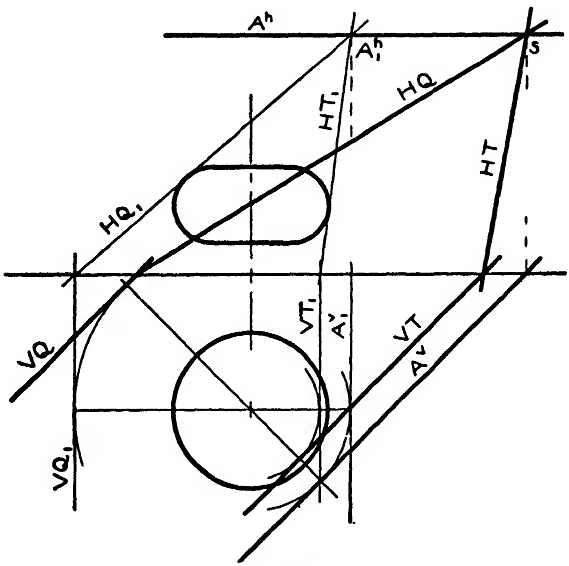

FIG. 302.

draw the tangent planes Q_1 (HQ_1, VQ_1) and T_1 (HT_1, VT_1). Revolve Q_1 and T_1 about the axis of the torus to the positions Q (VQ, HQ) and T (VT, HT), so that each plane contains the line A (§ 134). Then Q and T are the required tangent planes.

CHAPTER XXI

THE INTERSECTION OF CURVED SURFACES BY PLANES

172. Classification of Curved Surfaces. Curved surfaces may be divided into three classes:

(*a*) ***Ruled surfaces***, on which a straight line, or rectilinear element, can be drawn through any point of the surface;

(*b*) ***Surfaces of revolution***, on which a circle, lying in a plane perpendicular to the axis, can be drawn through every point of the surface;

(*c*) ***All other curved surfaces***, on which, in general, neither straight lines nor circles can be drawn.

Certain surfaces, for example the cone and cylinder of revolution, belong to both classes (*a*) and (*b*).

173. The Intersection of a Ruled Surface and a Plane. Select a sufficient number of the rectilinear elements of the surface. Find the points in which these straight lines intersect the given plane. These points lie in the required intersection.

In the process of finding where a straight line intersects a plane, it is usual to pass through the line an auxiliary plane perpendicular to H or V. (See Prob. 13, § 119.) Hence the elements of the ruled surface are often chosen by passing through the surface a series of planes perpendicular to H or V, these same planes being then used as auxiliaries for finding the points in which the resulting elements intersect the secant plane.

174. The Intersection of a Surface of Revolution and a Plane. An auxiliary plane taken perpendicular to the axis of the given surface will cut one or more circles from the surface, and, in general, a straight line from the given plane. Points of intersection, if any, of the straight line with the circle or circles, will lie in the required intersection of the plane and the surface.

In order that the circular sections of the given surface may be projected readily, the axis of the surface should be perpendicular to one of the coördinate planes. (Compare §§ 84, 86, 164.)

175. The Intersection of Any Curved Surface and a Plane. In general, the intersection of a surface and a plane can be found by drawing on the given surface any series of curves which are known to lie in the surface, and then finding the points in which these curved lines intersect the given plane.

If the curves which can be drawn in the surface are plane curves, the planes of these curves may be used as auxiliary planes, and the process of finding the intersection is similar to that for a surface of revolution.

176. Method by Means of Secondary Projections. Any plane may be seen edgewise by projecting on a suitable secondary plane of projection (§ 70). A general method of finding the intersection of any surface or solid with a plane is, therefore, to make a secondary projection of the surface and the secant plane so that the plane appears edgewise. This reduces one projection of the required intersection to a straight line. Since this method involves the construction of a new projection of the given surface, as well as of the secant plane, it will be found of doubtful advantage in the case of surfaces consisting of rectilinear elements. The method is, however, a good one with surfaces of revolution, whose projections, from any point of view perpendicular to the axis, are alike.

177. Visibility of the Curve of Intersection. In finding the intersection of a plane with a curved surface or solid, the plane will be considered to be transparent, and in general no portion of the surface or solid will be removed. The various portions of the resulting line of intersection, therefore, will be visible or invisible, according as they lie in a visible or an invisible portion of the surface. (Compare § 85.)

178. A Rectilinear Tangent to the Curve of Intersection. At any point in the intersection of any curved surface and a

plane, a rectilinear tangent to the intersection may be drawn without knowing or considering the properties of the curve of intersection. For the tangent line must lie in each of two planes: first, in the given secant plane, since every tangent to a plane curve lies in the plane of the curve (§ 153); and second, in the plane tangent to the given surface at the point in question (§ 154). Hence, if the plane tangent to the surface at the given point can be found, the line of intersection of this plane with the given secant plane will be tangent to the curve of intersection at the given point.

179. Development. A curved surface is developable when it can be unrolled into a plane, without extension, compression, or distortion of any kind. Only surfaces which consist of rectilinear elements are developable; and of these surfaces, only those forms in which every two consecutive elements lie in the same plane, that is, either intersect or are parallel. Cones and cylinders are developable, and are the only forms of developable curved surfaces commonly used in practical work.

CHAPTER XXII

INTERSECTION OF PLANES WITH CONES AND CYLINDERS

180. The Intersection of a Cone or Cylinder with a Plane. The cone and the cylinder are ruled surfaces. Hence the method of finding points in the intersection is that described in § 173.

In the three problems which follow, in addition to finding the curve of intersection we shall draw a tangent line to some point of the intersection (§ 178), and develop the curved surface, showing the section and the tangent line.

Problem 41. ***To find the intersection of a cone and plane.***

CASE I. *The base of the cone lies in H or V.*

A. THE INTERSECTION. **Analysis.** See § 173.

Construction (Fig. 303). The base of the given cone lies in *H*. The given secant plane is *Q*. Eight elements have been chosen on the cone, distinguished by the numbers 1 to 8 around the base. Find the points in which these elements intersect the plane *Q* by passing through them auxiliary planes perpendicular to *H* or *V* (Prob. 13, Usual Method, § 119). The planes here chosen are the planes *X*, *Y* . . . *N*, perpendicular to *H*. The intersection of planes *X* and *Q* is the line *A*. The intersection of A^v and the *V*-projection of the element 0–3 locates 13^v, one point of the required intersection. The intersection of planes *Z* and *Q* is the line *C*, the projection C^v being parallel to *VQ*. Since plane *Z* contains two elements of the cone, the construction locates two points, 11 on element 0–1 and 15 on element 0–5. But the planes *Y*, *M*, and *N* are so situated that in attempting to find their lines of intersection with *Q*, only one point in each line can be found within the limits of the figure, namely, the points b^v, d^v, and e^v on the ground line.

B. The Common Point. Since the planes *X*, *Y*, *Z*, *M*, and *N* are perpendicular to *H*, it follows that the lines of inter-

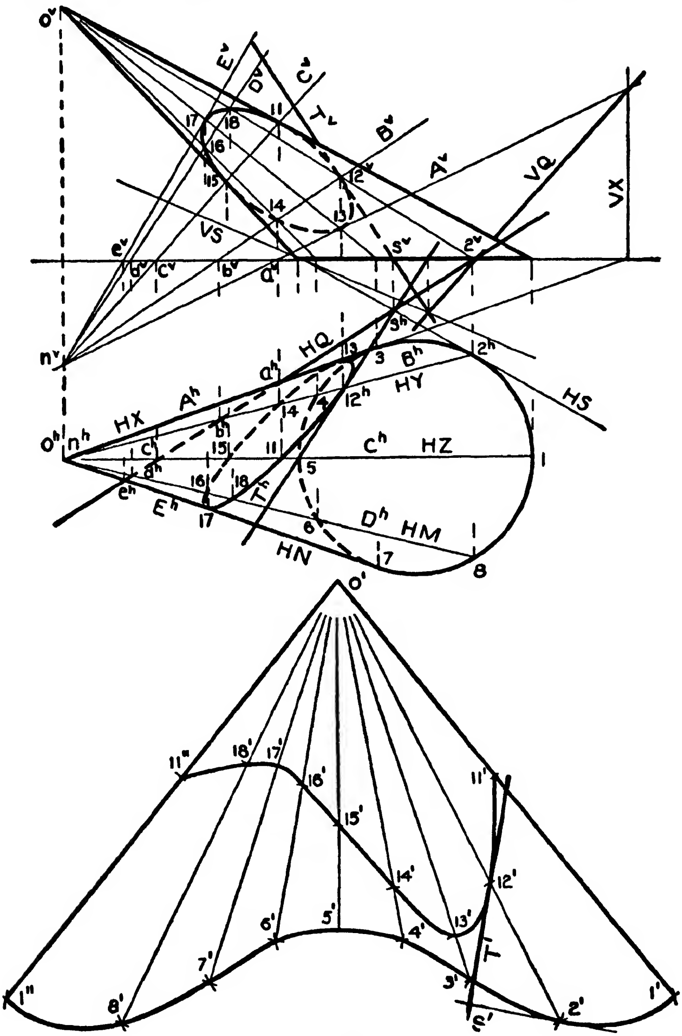

Fig. 303.

section *A*, *B*, *C*, etc., will have their *H*-projections coincident with *HX*, *HY*, *HZ*, etc., respectively. But the lines of inter-

section A, B, C, etc., are all lines in one plane, namely, the plane Q. Now if a number of lines known to be in the same plane appear to intersect in a common point in one view, it can only be because the lines actually intersect in space. Hence in the other view the lines must also pass through a common point. In the figure, the H-projection of this point is n^h, while n^v lies on A^v. The other V-projections, B^v, D^v, and E^v, may now be drawn through n^v and the points b^v, d^v, and e^v on GL, already determined. The intersections of B^v, D^v, and E^v with the elements lying in the corresponding auxiliary planes locate the remaining points in the curve of intersection.

C. Visibility of the Curve of Intersection. (See § 177.) In the H-projection, the point 11, on the highest element, is evidently visible. This determines the visible side of the curve, which will become invisible at the points 13 and 17, which lie on the boundaries of the visible surface of the cone. Similarly, in the V-projection, the point 17, on the extreme front element, is visible. The visible side of the curve, beginning at 11, passes through 17 and ends at 15.

D. A Line Tangent at a Given Point in the Curve of Intersection. **Analysis.** See § 178.

Construction. Let 12, lying on the element 0–2, be the given point. Find the plane S (HS, VS), which is tangent to the cone at the point 12 (Prob. 31, § 162). Find the intersection of S with the given plane Q (Prob. 12, § 118). This line of intersection T (T^h, T^v) is the required tangent line. As a check, it should pass through the point 12, and show tangent to the curve of intersection in each projection (§ 152).

E. Development of the Curved Surface of the Cone. **Analysis.** The entire curved surface, between the vertex and the base, should be developed first. Any other line drawn on the surface can then be put in. The conical surface is divided by its elements into pieces bounded by two straight lines and a curve. These pieces are approximately plane triangles, the approximation being closer the nearer the elements are together. The curved surface may then be developed as if the

cone were a many-sided pyramid, the triangular faces being built up from the true lengths of their sides. (See § 89.)

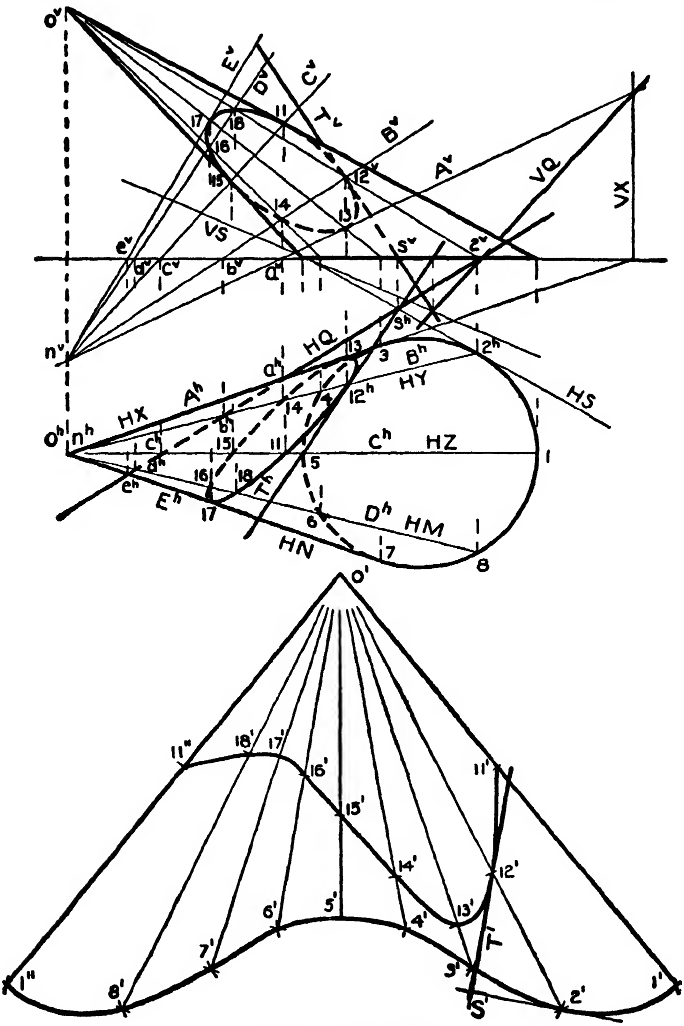

Fig. 303 (repeated).

Construction. The given conical surface is divided into eight triangular pieces by the chosen elements 0–1, 0–2 . . .

0–8. The true lengths of the elements may be found by the rule of § 88. The true lengths of 1–2, 2–3, etc. appear directly in the H-projection of the base, the usual approximation being that the chord equals the arc. Assuming the true lengths found as needed, begin the development by laying off, on any convenient line, $0'$–$1'$ equal to the true length of 0–1. With $0'$ and $1'$ as centers, radii respectively the true lengths of 0–2 and 1–2, strike arcs intersecting at $2'$. Working from $0'$ and $2'$, obtain $3'$ in a similar manner, and so on. Draw a smooth curve through $1', 2' \ldots 8', 1''$.

To locate the section 11, 12 . . . 18, 11 in the development, note that $11'$ may be located by measuring from $0'$ the true length of 0–11, or from $1'$ the true length of 1–11. By comparing Figs. 113 and 114, § 88, it will be seen that in this case the true length of 1–11 is the more easily found. To continue, locate $12'$ from $2'$ by using the true length of 2–12, $13'$ from $3'$ by the true length of 3–13, and so on. Draw a smooth curve through the points $11', 12', 13' \ldots 18', 11''$.

F. THE TANGENT LINE IN THE DEVELOPMENT. **Analysis.** The tangent line at any point in the section makes a definite angle with the element passing through that point. This angle must appear in its true size in the development.

Construction. The line T intersects the element 0–2 at the point 12, and makes with it the angle 2–12–s. Let us connect the point 2 on the element 0–2 with the H-trace, s, of the tangent line T. The line is already drawn, the H-projection, 2^h–s^h, being a portion of the H-trace, HS, of the tangent plane at the point 2, while 2^v–s^v lies in GL. We thus have found a plane triangle, 2–12–s, one of whose angles is the required angle 2–12–s. Points 2 and 12 are already in the development at $2'$ and $12'$, respectively. Find the true lengths of 12–s and 2–s. Using $12'$ and $2'$ as centers, strike arcs intersecting at s'. Then $2'$–$12'$–s' is the true size of the angle 2–12–s. Consequently s'–$12'$ is the required development, T', of the tangent line T. As a check, it should be tangent to the curve $11'$–$12'$–$13'$. Incidentally, since the line s'–$2'$ is the

development of a line tangent to the base, s'–$2'$ is tangent to the curve $1'$–$2'$–$3'$.

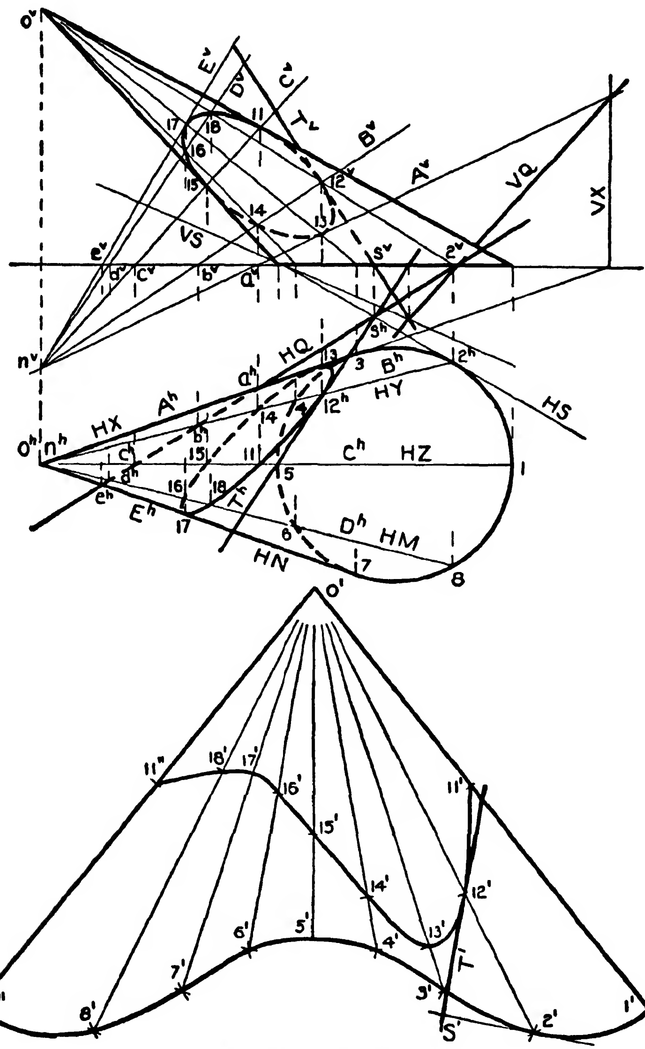

FIG. 303 (repeated).

It is geometrically possible to lay off the triangle $2'$–$12'$–s' on the other side of $2'$–$12'$. But by visualization of the posi-

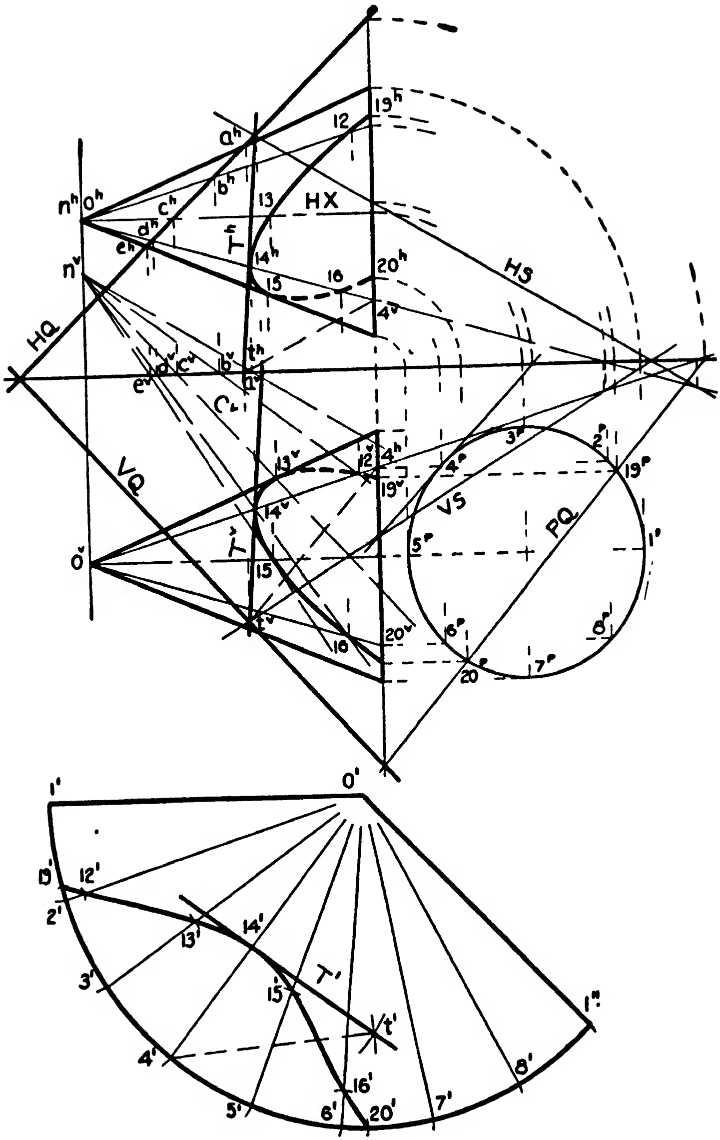

FIG. 304.

tion of the point s relative to the cone, as shown especially in the H-projection (s on the side of element 0–2 *away* from 0–1) it will be seen that the position of s' as given in the development is the only one which represents the point.

The remaining cases of this problem differ in details of construction, but do not differ in essential principles, from the first case. They will therefore not be taken up so fully, but all important variations will be noted.

CASE II. *The base of the cone lies in P.*

Construction (Fig. 304). The given cone is one of revolution lying in the third quadrant. The given secant plane is Q. Eight elements have been chosen, equally spaced on the P-projection of the base.

The points in which the chosen elements intersect the plane Q are found by means of auxiliary planes perpendicular to H. The plane X, passed through the element 0–3, intersects Q in in the line C, the V-projection C^v being parallel to VQ. The V-projection, n^v, of the common point is found where C^v intersects the projector $o^h o^v$. The remaining lines of intersection are drawn through n^v and the points a^v, b^v, d^v, e^v, on GL. As soon as these lines are drawn, it is seen that some of the points of the intersection lie beyond the limits of the cone.

TO FIND THE POINTS WHERE THE CURVE OF INTERSECTION LEAVES THE BASE. Find the profile trace, PQ, of Q (§ 60). Since this is a line in the plane of the base, the intersections of PQ with the profile view of the base, namely 19^p and 20^p, are the profile projections of two points in the curve of intersection.

The line T is tangent to the intersection at point 14. This line is the intersection of the given plane Q with the plane S, the latter being passed tangent to the cone at the point 14.

The development, $0'$–$1'$–$2'$–$3'$ . . . $8'$–$1''$–$0'$, of the entire curved surface is the sector of a circle, since all the elements are of the same true length. The lengths $1'$–$2'$, $2'$–$3'$, etc., are equal to 1^p–2^p, 2^p–3^p, etc. Such a point as $13'$ may be located equally well by making $3'$–$13'$ equal to the true length of 3–13, or by making $0'$–$13'$ equal to the true length of 0–13.

The point 19′ is found by making 2′–19′ equal to 2^p–19^p. The point 20′ is found by making 6′–20′ equal to 6^p–20^p.

To develop the tangent line *T*, connect some point on *T* with some point on the element 0–4, thus forming a triangle. The points chosen are the *V*-trace, *t*, of *T*, and the extremity, 4, of the element. The triangle 4–14–*t* is then plotted in the development at 4′–14′–*t*′, by using the true length of its sides. Then *t*′–14′ is the development of *T*, and is tangent to 13′–14′–15′.

But since *t*–4 was not tangent to the base, the development *t*′–4′ will not be tangent to 3′–4′–5′.

Case III. *The base of the cone does not lie in H, V, or P.*

Construction (Fig. 305). The method of finding points in the intersection does not differ from that previously explained. We may note, however, that the auxiliary planes are taken here perpendicular to *V*. Consequently the *V*-projection n^v of the common point coincides with o^v, while n^h is found on the projector $o^v o^h$.

The line *T*, tangent at the point 16, is the intersection of *Q* with the plane *R*, tangent at point 16.

The development requires the true lengths of the elements, and the true lengths of the segments 1–2, 2–3, etc., of the base. Since neither end of any element lies in a coördinate plane, the true lengths of the elements must be found by the method of Fig. 113, § 88.

The true size of the base does not appear in either projection, and must be found before the true lengths of its segments become known.

The true size is found here by revolving about the diameter 1^v–5^v, which is parallel to *V*. Then the true size is shown by the points 2_r, 3_r, etc. Therefore, in the development, 1′–2′ equals 1^v–2_r, 2′–3′ equals 2_r–3_r, etc.

The developed tangent is obtained from the triangle 6–16–*s*, the points 6 and *s* being chosen as convenient points on the element and tangent line, respectively. The development *T*′ is tangent to 15′–16′–17′, but the development *s*′–6′ is not tangent to 5′–6′–7′.

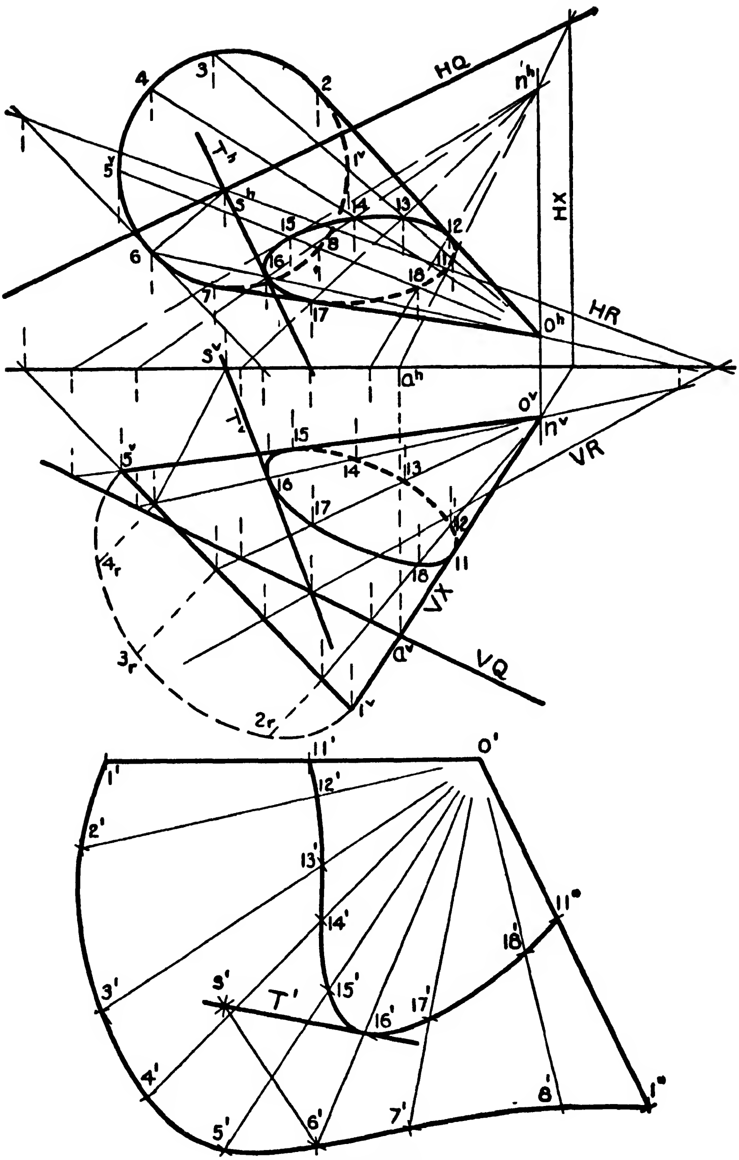

FIG. 305.

Problem 42. *To find the intersection of the frustum of a cone and a plane.*

Analysis. See §§ 173, 177. The general principles involved are the same as in the preceding problem.

Construction (Fig. 306). The given frustum is placed in the first quadrant, with its lower base on H, its upper base parallel to H. The given secant plane is Q.

A. TO LOCATE THE ELEMENTS. Contour elements like 1–11 and 4–14 are readily obtained by noting certain tangent points. The arcs 1^h–4^h and 11^h–14^h may then be divided in any equal ratios by the points 2^h, 3^h, 12^h, 13^h, which will determine the elements 2–12, 3–13. Or, we may note that the plane J, perpendicular to H, which contains the element 5–15, must also contain the element 3–13.

The frustum in Fig. 306 has been specially chosen. It is not in general possible to project eight elements with so few lines in each view.

B. THE AUXILIARY LINE. Let us attempt to find the points in which the eight chosen elements intersect the plane Q, by passing through them planes perpendicular to H. Take for example the plane X, containing the element 8–18. The intersection of HX and HQ locates one point, a, in the line of intersection of these planes. A second point cannot be located by the use of VX. A similar situation is found for the planes Y, Z, etc. Even if some of the lines of intersection could be found by means of the V-traces of the planes, it would not assist in finding the other lines of intersection, since the common point of the preceding problem is not available. In this case, had we begun by taking the auxiliary planes perpendicular to V, we would have been confronted by a similar situation.

The method given below is a general one, and may be used when finding the intersection of any ruled surface with a plane.

Let L (L^h, L^v) be any arbitrarily chosen line lying in the plane Q. For convenience, this line is usually taken as one of the principal lines (§ 99) of the plane. Other conditions gov-

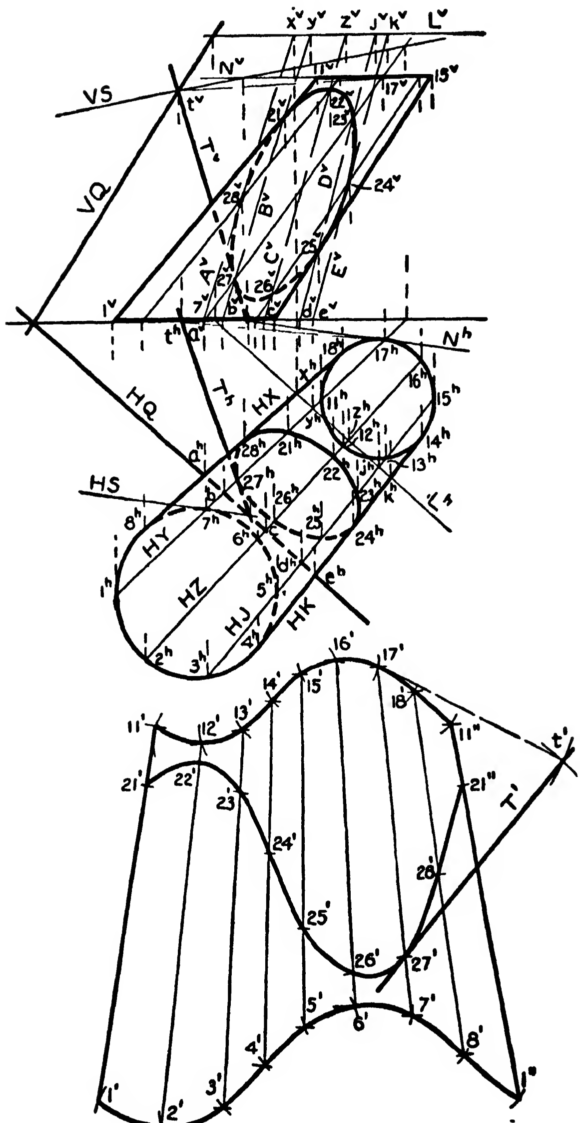

FIG. 306.

erning the choice of the line L should be apparent after observing the use which is made of it. The line L intersects the plane X in the point x (see Fig. 178, § 119). But L lies in the plane Q. Hence the point x lies in both the planes X and Q, consequently is a point in their line of intersection, A. The V-projection, A^v, must therefore pass through x^v in L^v, and is determined by x^v and the point a^v, previously noted in the ground line. Similarly, the line L intersects the plane Y in the point y, which must lie in the line of intersection of Y and Q; and so on. The V-projections of the lines of intersection of the auxiliary planes with Q being thus determined, the points on the curve of intersection are noted as in the preceding problem.

C. A LINE TANGENT TO THE CURVE OF INTERSECTION. **Analysis.** See § 178.

Construction. The point chosen is 27, lying on the element 7–17. The plane S is passed tangent to the frustum at this point (Prob. 31, § 162). The required tangent line, T, is the line of intersection of S with the given plane Q.

D. DEVELOPMENT OF THE CURVED SURFACE. **Analysis.** The portion of the curved surface between any two adjacent elements is approximately a plane quadrilateral, and is assumed to be such. Any quadrilateral is divisible by either of its diagonals into two plane triangles, the true sizes and shapes of which may be plotted from the true lengths of the sides.

The above method of development is a general one, and is known as **development by triangulation.** For the most accurate results, each quadrilateral should be divided into triangles by using the shorter diagonal.

Construction. In any convenient position, lay off 1′–11′ equal to the true length of 1–11. Triangulate 12′ from 1′ and 11′, using the true lengths of 1–12 and 11–12. The true length of 1–12 may be found with the compasses without actually drawing either projection of the line, while the true length of 11–12 is 11^h–12^h. Triangulate 2′ from 1′ and 12′, just located, using the true lengths of 1–2 and 2–12. Similarly, triangulate 3′ from 2′ and 12′, then locate 13′ from 3′ and 12′, and so on until

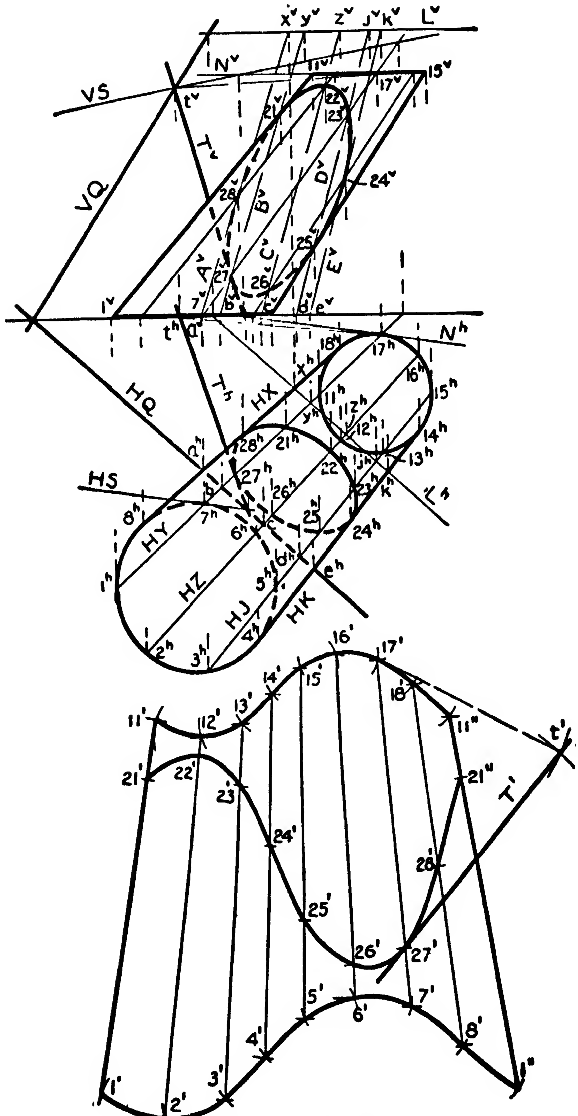

Fig. 306 (repeated).

the development is complete. Draw the developed elements 2′–12′, 3′–13′, etc., as fast as they are obtained, and note as a check whether they appear to converge to a point as the elements of a correctly developed cone should do.

E. The Development of the Curve of Intersection. The points 21′, 22′, 23′, etc., of the curve of intersection are here found by making 1′–21′ equal to the true length of 1–21, 2′–22′ equal to the true length of 2–22, and so on. They might also be found by making 11′–21′ the true length of 11–21, 12′–22′ that of 12–22, etc., but in this case the former set of true lengths are the easier to obtain. Why?

F. The Development of the Tangent Line. Note the triangle 27–*t*–17, one of whose sides lies along the tangent line *T*, and a second side along the element 7–17 containing the given point 27.

Triangulate *t*′ from 17′ and 27′, using the true lengths of 17–*t* and 27–*t*. Then *t*′–27′ is the required development, *T*′, of the tangent line *T*.

Problem 43. ***To find the intersection of a cylinder and a plane.***

Analysis. See §§ 173, 177, 178. The same general principles apply as in the preceding two problems.

Case I. *The base of the cylinder lies in H or V.* **Construction** (Fig. 307). The base of the given cylinder lies in *H*. The given cutting plane is *Q*. Eight elements of the cylinder have been chosen. The points in which these elements intersect the plane *Q* are here found by passing through them auxiliary planes perpendicular to *V*.

The construction is practically a repetition of that of the preceding problems. We should note, however, that since the elements of the surface are parallel, the auxiliary planes through these elements are parallel. Hence the lines of intersection, A^h, B^h, etc., are parallel.

This gives a method of drawing some of these lines when but one point of the line is available. The resulting intersection is the curve 11–12–13 . . . 18–11.

The visibility of the section is obtained as explained in § 177.

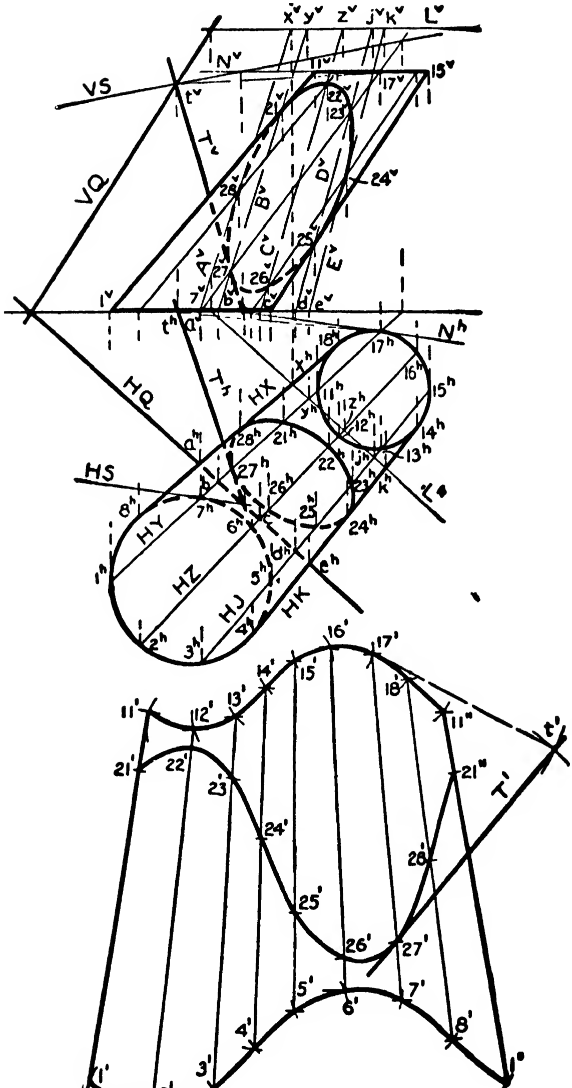

FIG. 306 (repeated).

The line T (T^h, T^v) is tangent to the section at the point 16. This line is the intersection of the plane S, passed tangent to the cylinder at the point 16 (Prob. 34, § 162) with the given plane Q (§ 178).

DEVELOPMENT OF THE CURVED SURFACE BETWEEN THE BASE AND SECTION. **First Analysis.** The portion of the curved surface bounded by two adjacent elements and the included portions of the base and section is approximately a plane quadrilateral (a trapezoid). The method of development by triangulation used to develop the frustum of the cone, Prob. 42, may therefore be employed.

Second Analysis. Obtain a right section of the cylinder, that is, a section made by a plane at right angles to the elements. Find the true size of the right section. The development of the right section is a straight line, whose length equals the circumference of the right section. The developed elements will be lines at right angles to the developed right section. Any desired points on the elements can be located by using the true distances from the right section.

In practical applications of the development of cylinders, the right section is usually known, being commonly a circle. Consequently development by means of the right section will be the method adopted here.

Construction. Any plane at right angles to the elements will cut a right section, and if no other condition prevails the plane may be chosen at random. Such a plane is the plane R. How are HR and VR drawn? (§ 112.)

The intersection of R and the cylinder must now be found. The construction is similar to, and entirely independent of, that used for finding the intersection of plane Q. The resulting section is 21–22–23 . . . 28–21.

Since this is a figure lying in plane R, its true size may be found by revolving about either trace of R, say about HR into H (Prob. 24, Cor. 1, § 142; Prob. 21, Working Rule, § 138). The resulting true size is shown at 21_r–22_r– . . . 28_r–21_r.

To begin the development, draw a straight line, 21′–21″, in

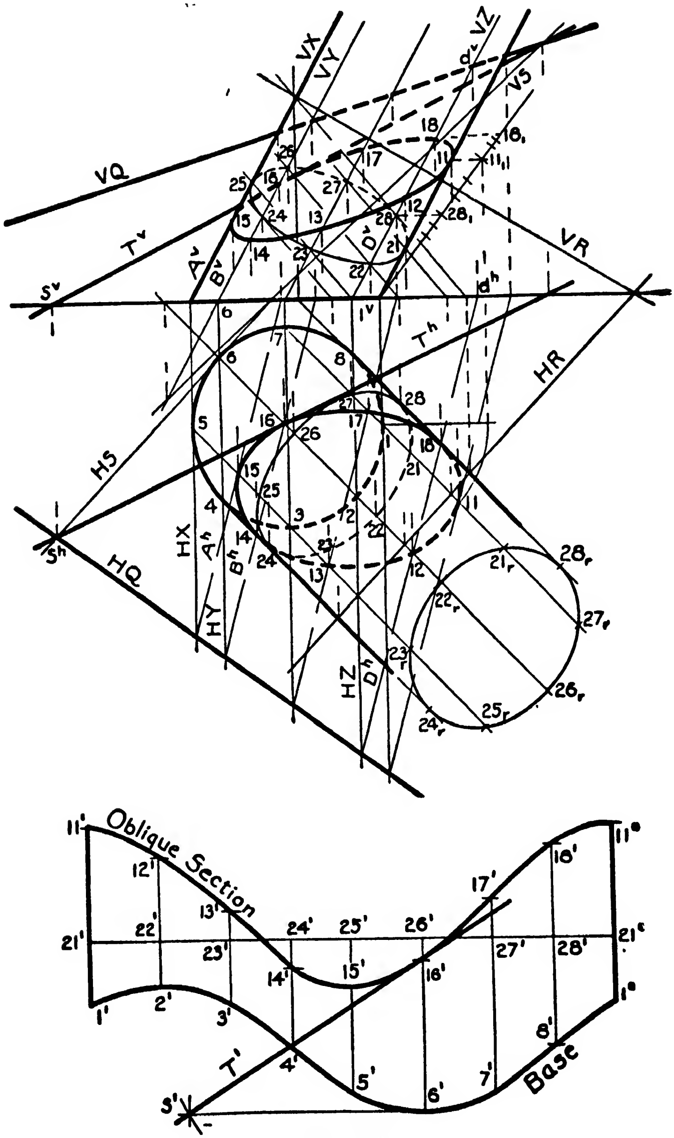
VX
VY
VZ
VS
VQ
VR
T^v
A^v
B^v
D^v
S^v
1^v
d^v
d^h
T^h
HR
HS
HQ
HX
HY
HZ
A^h
B^h
D^h
S^h
Oblique Section
Base
T'
S'

Fig. 307.

any convenient position. Make 21′–22′ equal to 21_r–22_r; 22′–23′ equal to 22_r–23_r; and so on up to 28′–21″ equal to 28_r–21_r. This straight line is the developed right section.

Draw perpendiculars, representing elements, at 21′, 22′, etc. Make 21′–1′ equal to the true length of 21–1; 22′–2′ equal to the true length of 22–2; etc. The curve 1′–2′–3′ . . . 8′–1″ thus obtained is the development of the base 1–2–3 . . . 8–1 of the cylinder.

To obtain the development 11′–12′–13′ . . . 11″ of the section made by the plane Q, we may make the distance 21′–11′ equal the true length of 21–11, 22′–12′ the true length of 22–12, etc.

Otherwise, we may make 1′–11′ equal to the true length of 1–11, 2′–12′ the true length of 2–12, etc. The latter method is preferable, as there is less chance for error. For example, compare the elements 3′–13′ and 4′–14′. If 13′ is measured from 3′ and 14′ from 4′, the direction of measurement is the same; but if 13′ is measured from 23′ and 14′ from 24′, the direction must be reversed.

A construction for obtaining rapidly all the true lengths of the elements required in making the development is shown in the figure. Let the true length of one element, for example 1–11, be obtained by revolving the line parallel to V, as shown at 1^v–11_1 (Prob. 13, First Method, § 78). Then, since all the elements are parallel, the true length of any element can be found by projecting to the line 1^v–11_1, produced if necessary. Thus, the true length of 8–28 equals 1^v–28_1, and the true length of 28–18 equals 28_1–18_1.

The developed tangent line, T', is determined from the triangle 6–16–s. The point s' is located by using its true distances, s'–16′ and s'–6′, from 16′ and 6′ respectively.

Case II. *The base of the cylinder lies in* P.

Construction (Fig. 308). The given cylinder is a cylinder of revolution lying in the third quadrant. It is intersected by the plane Q.

Eight elements have been chosen, indicated by the numbers 1 to 8 on the P-projection. Points 12, 13, 14, 15 of the intersec-

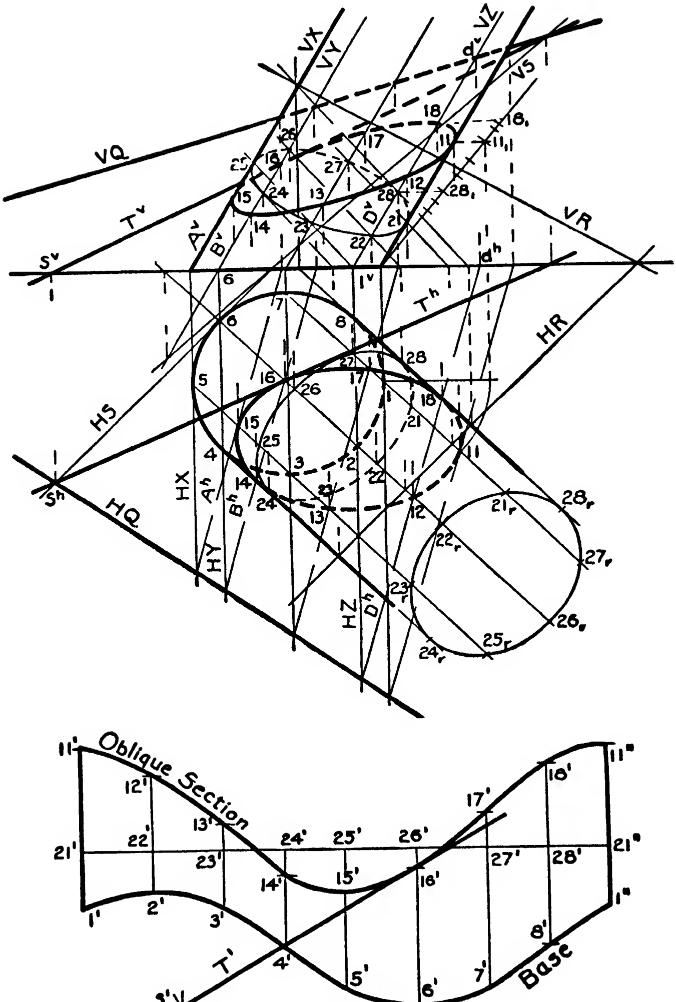

FIG. 307 (repeated).

tion, which lie within the limits of the cylinder, are found, as in the preceding examples, by finding where the chosen elements intersect the plane Q.

The plane Q intersects the left-hand base of the cylinder in the points 19 and 20. To find these points directly, as in the corresponding case of the cone, Fig. 304, note first that the profile trace of Q on the plane of this base would be determined from the points x and y, on HQ and VQ respectively (§ 60).

The base which is already projected in profile is the right-hand base. Therefore project x and y to x_1 and y_1, and obtain the projected trace, P_1Q. The points 19^p and 20^p, in which P_1Q intersects the circle, are the profile projections of the required points, which must then be projected back to the left-hand base.

The line T is tangent to the section at the point 14. This line is the intersection of the given plane Q with the plane S, which is passed tangent to the cylinder at point 14.

The development of the cylinder is obtained readily, since each base of the cylinder is a right section. The line $1'$–$2'$–$3' \cdots 8'$–$1''$ is the development of the right-hand base, the distances $1'$–$2'$, $2'$–$3'$, etc. being equal to 1^p2^p, 2^p3^p, etc. By placing the development of this base in line with its projections, as shown, the points $12'$, $13'$, $14'$, $15'$ may be obtained by projection. The points $19'$ and $20'$ are obtained from the positions of 19^p and 20^p.

The development, T', of the tangent line is obtained by locating in the development the triangle 4–14–t.

Case III. *The base of the cylinder does not lie in H, V or P.*

Construction (Fig. 309). The given cylinder lies in the third quadrant, and is intersected by the plane Q in the curve 11–12–13 $\cdots$ 18–11. The construction involves nothing that has not been already explained.

The line T, tangent at the point 12 of the curve of intersection, is the line of intersection of the given plane Q with the plane S, passed tangent to the cylinder at point 12.

The development is obtained by the use of a right section. This is the section 21–22–23 $\cdots$ 28–21, cut by the plane R, per-

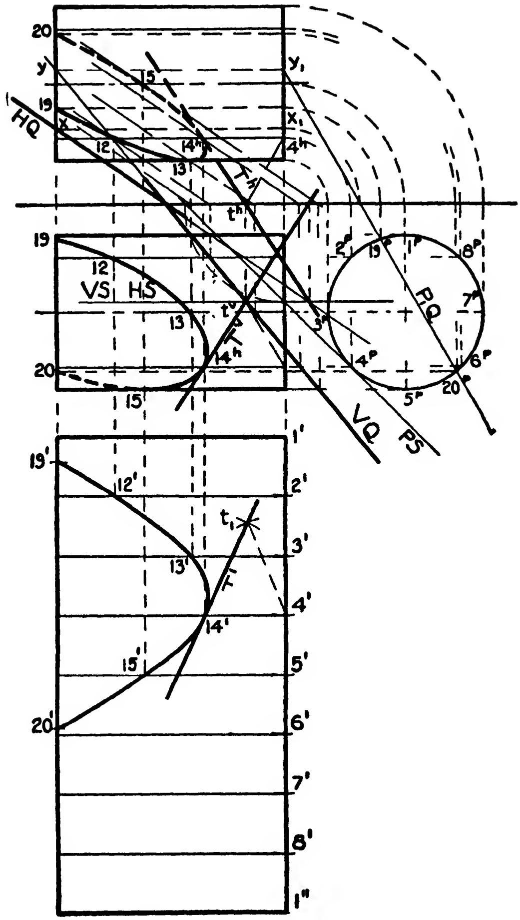
20
y
19
X
HQ
15
y1
X1
12
14h
4h
13
Th
th
19
12
VS
HS
13
tv
Tv
14h
20
15
2p
19p
1p
8p
PQ
7p
3p
4p
6p
5p
20p
VQ
PS
1'
19'
12'
2'
t1
3'
13'
T1
4'
14'
15'
5'
20'
6'
7'
8'
1''

FIG. 308.

pendicular to the elements, but otherwise arbitrarily chosen. The true size of the right section, obtained by revolving about *VR* into *V*, appears at 21_r–$22_r \cdots 28_r$–21_r, and its rectification or development at $21'$–$22' \cdots 28'$–$21''$. The true lengths of the elements are obtained by projecting them all on to the true length of 5–15, obtained at 5^v–15_1 by using the method of Prob. 3, § 80.

The development of the tangent line is obtained by plotting the triangle 2–12–*c*, the point *c* being any arbitrary point on the line *T*. In determining the triangle which shall be used to develop the tangent line, one side must necessarily be a portion of the tangent line itself, and a second side must be a portion of the element. The third side of the triangle may be a line connecting any convenient points on each of the other two lines. Heretofore, the point chosen on the tangent line has always been one of its traces. This is not necessary, however, and in this case a better shaped triangle and consequently more accurate result is obtained by taking a different point.

181. The Intersection of a Pyramid or Prism with a Plane. Although the pyramid and prism are bounded by plane instead of curved surfaces, the intersection of either of these solids with a general oblique plane can often be found most advantageously by the method of the preceding Article. Consequently the following problem may logically be introduced at this time.

Problem 44. ***To find the intersection of a pyramid or prism and a plane.***

Analysis. Let a pyramid be given. Consider the lateral edges as the elements of a cone. Find the points in which these elements intersect the given plane, using the method of Prob. 41 or 42, § 180. Connect the points thus found by straight lines.

Let a prism be given. Consider the parallel lateral edges as the elements of a cylinder, and use the method of Prob. 43, § 180. Connect the points found by straight lines.

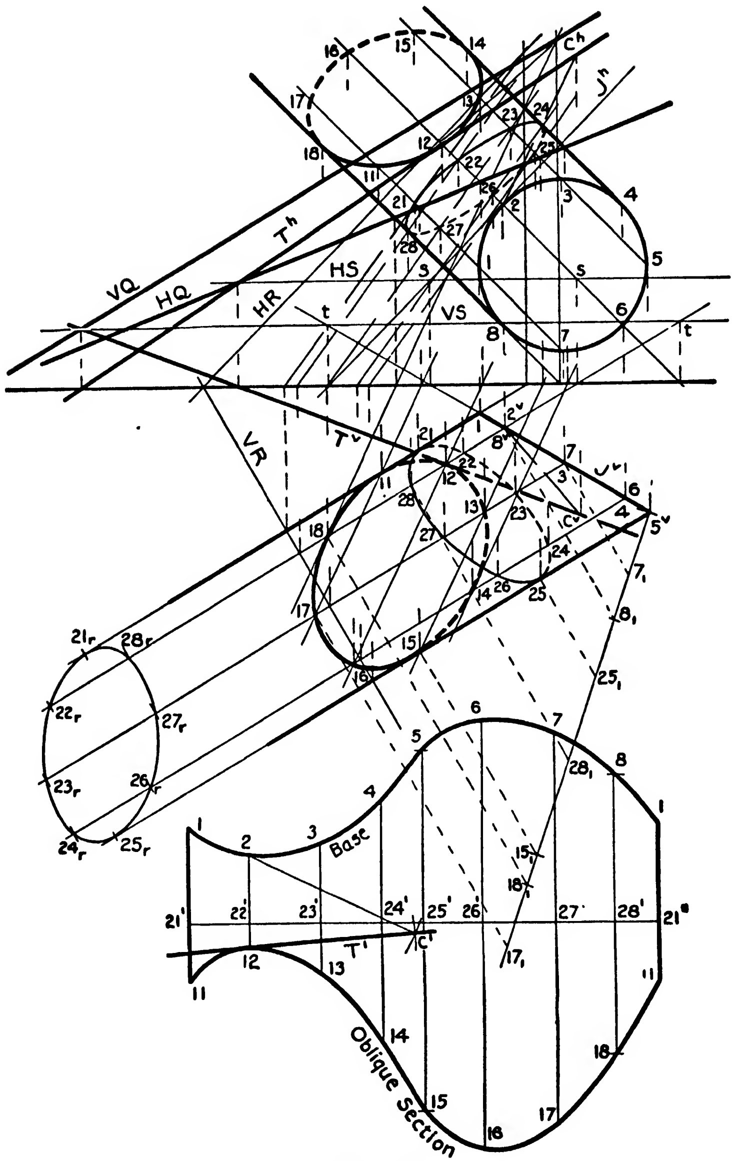

FIG. 309.

Construction (Fig. 310). The given solid is a pyramid. The points in which the lateral edges 0–1, 0–2, 0–3, 0–4 intersect the given plane Q are found by auxiliary planes X, Y, Z,

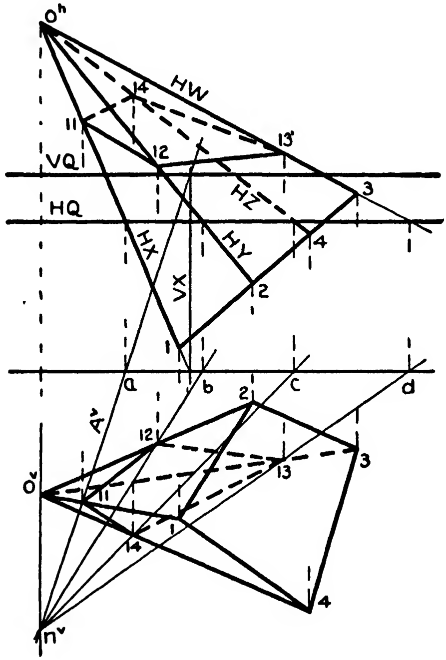

FIG. 310.

W perpendicular to H. The plane X (HX, VX) locates the common point n^v, after which the V-traces of the remaining planes are not needed. The required intersection is the quadrilateral 11–12–13–14.

CHAPTER XXIII

INTERSECTION OF PLANES WITH SURFACES OF REVOLUTION

182. The Intersection of a Surface of Revolution and a Plane. The general method of obtaining points in the intersection is by means of auxiliary planes perpendicular to the axis of the surface, as has already been explained in § 174. Besides the points thus found, other points, lying on particular planes that contain the axis of the surface, are usually essential.

The method outlined above applies to any surface of revolution, not exclusively to double curved surfaces. We shall, however, apply it in the present chapter only to such surfaces.

With certain double curved surfaces of revolution, noticeably the torus, the use of a secondary projection (§ 176) is often advantageous. This method does not affect the general principles, but makes these principles easier to apply.

Problem 45. ***To find the intersection of a double curved surface of revolution and a plane.***

Analysis. See §§ 174, 176, and above.

First Construction (without the use of a secondary projection) (Fig. 311). The given surface is that of a circular spindle, formed by the revolution of the arc of a circle about its chord. The axis of the surface is placed perpendicular to H. An auxiliary plane perpendicular to the axis A, such as X(VX; there is no HX) intersects the given surface in the circle C, a parallel of the surface (§ 166), and the given plane Q in the line B (Prob. 12, Special Case III, § 118). The intersection of B^h and C^h determines the H-projections of two points in the required intersection. Other points are similarly obtained by other auxiliary planes perpendicular to the axis A. In the figure are shown the planes X_1, symmetrical with X, which gives the points 3 and 4, and the plane Z of the greatest parallel E, which locates the points 5 and 6.

A. Visibility of the Curve of Intersection. In the H-projection is seen all that part of the surface which lies above the plane Z of the greatest parallel. The points which divide the visible from the invisible portion of the curve are 5^h and 6^h, already determined as in this plane. In the V-projection appears the half of the surface in front of the principal meridian plane Y (HY, there is no VY; § 166). The points of division between the visible and invisible parts of the V-projection of the intersection lie in the plane Y, and should be obtained by the direct use of this plane as an auxiliary.

B. Points Determined by Meridian Planes. The plane Y intersects the given surface in the meridian section which forms the outline of the V-projection. The plane Y intersects the given plane Q in the line J, whose V-projection, J^v, intersects the meridian in the points 7^v and 8^v. These are points in the curve of intersection, and, as previously noted, are the points of division between the visible and invisible portion of the curve in this projection.

An inspection of the H-projection shows that the intersection is symmetric with respect to the meridian plane M (HM, VM), which is perpendicular to Q. For this reason, the plane M is called the **meridian plane of symmetry**. The points of the curve which lie in M are important, and should be found by using M as an auxiliary. In this case they are the highest and lowest points of the curve. In any case they are points of maxima or minima, at which the tangent to the curve lies in a plane perpendicular to the axis of the surface.

The plane M intersects the given surface in a meridian (§ 165), the counterpart of that appearing as the V-projection of the surface. This meridian projects on H as a portion of HM; it is not projected on V, as the need of this projection can be avoided. The plane M intersects the given plane Q in the line K (K^h, K^v) (Prob. 12, § 118). Since the plane M contains the axis, A, of the surface, the lines A and K intersect in a point, o, whose V-projection is at o^v. Note that J^v, previously determined as lying in the principal meridian plane

Y, must also pass through o^v, the point o being in fact the intersection of the axis A with the given plane Q.

Revolve the plane M about the axis A until M coincides with the principal meridian plane Y. The line K, lying in M, will take the position K_r (K_r^h, K_r^v). The revolution is here

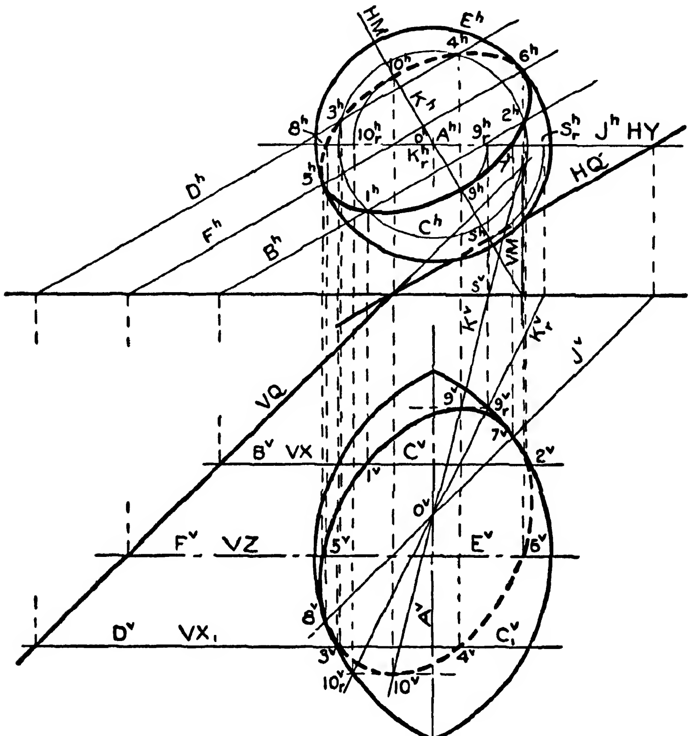

Fig. 311.

effected by using the H-trace, s, of K, and the fixed point, o, on the axis A. The meridian lying in M will coincide with the principal meridian, and therefore project on V as the contour of the surface. The intersection of K_r^v with this contour at 9_r^v and 10_r^v determines the revolved positions of two points in the curve of intersection. The actual projections, 9^v, 10^v, 9^h, and 10^h, result from counter-revolving the line K.

Second Construction (using a secondary projection) (Fig. 312). The given torus is to be intersected by the plane Q. To obtain an edge view of Q, assume a secondary ground line,

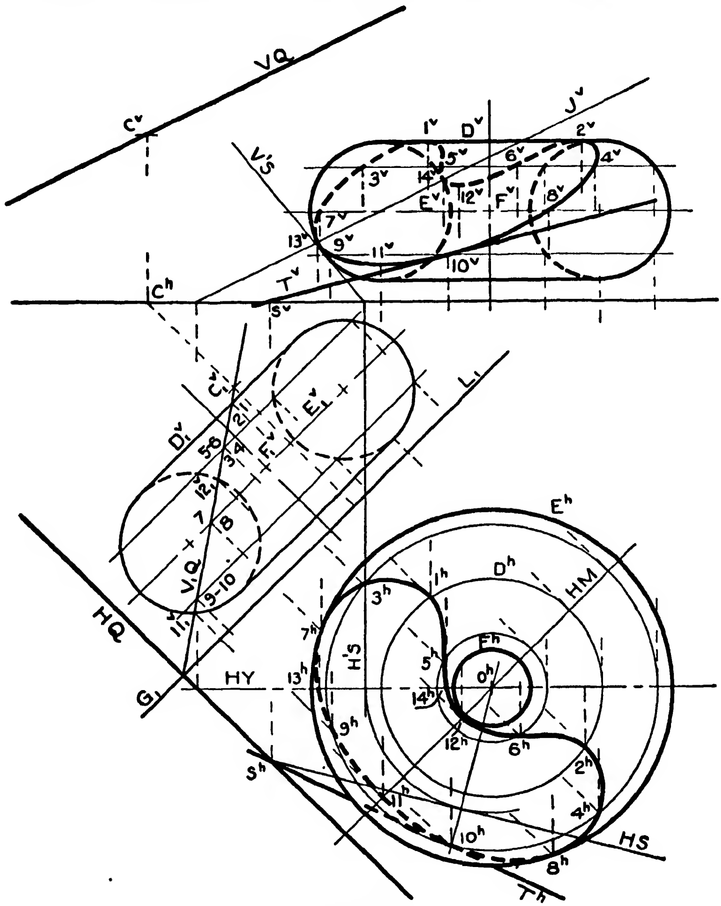

FIG. 312.

G_1L_1, perpendicular to HQ in any convenient location; then proceed as in § 70. The secondary projection of the given torus is identical in form with the original V-projection.

Considering now the H-projection and the secondary V-projection as the given projections of the torus, and V_1Q as

the edge view of the secant plane, we have the same conditions as in §§ 84, 85, and 86; in particular, a case similar to that shown in Fig. 109. Having obtained the points 1 to 10 in the *H*-projection by the construction there given, the same points are projected to the original *V*-projection (§§ 68, 72).

A. Points Determined by Meridian Planes. Points on the meridian plane of symmetry *M* are needed to complete the *H*-projection. These and the points on the principal meridian plane *Y* are needed to complete the *V*-projection.

The meridian of the torus which lies in the plane of symmetry *M* appears in true size in the secondary *V*-projection, it being the two circles which form a part of this projection. Since V_1Q is an edge view, the points required appear directly at 11_1^v and 12_1^v, where V_1Q intersects one of these circles. The *V*-projections, 11^v and 12^v, are found by transferring heights from the secondary projection.

The points in the principal meridian plane *Y* are found as explained in the preceding case. The plane *Y* intersects *Q* in the line *J*, the *V*-projection J^v being parallel to *VQ*. The meridian projects as the two circles which are a part of the *V*-projection. The possible intersections, 13^v and 14^v, of J^v with these circles determine the required points. At 13^v and 14^v the projection of the curve of intersection is tangent to the meridian circle.

B. Visibility of the Curve of Intersection. In the *H*-projection can be seen the upper half of the surface, lying above the plane of the greatest and least parallels. The points visible are 1, 2, 3, 4, 5, 6, 7, 8, 12. In the *V*-projection, only the front half of the outer portion of the surface can be seen. This visible portion is projected in the plan as the part in front of *HY* and outside of circle *D*. A point like 6, for example, although in front of the principal meridian plane *Y*, is nevertheless invisible in *V*-projection, being hidden by a portion of the surface still further in front. Beginning at the left, the visible portion of the *V*-projection begins at point 13, lying in the principal meridian plane *Y*, extends through 9, 11, 10, 8, 4, and ends at point 2 in the highest parallel, *D*.

C. A Line Tangent to the Intersection. A tangent line is drawn at the point 10. This line is the intersection of the given plane *Q* with the plane *S*, tangent to the surface at point 10 (§ 178). Find the tangent plane *S* by any convenient method (Prob. 38, § 170). Here only the *H*-trace, *HS*, is found. The intersection of *HS* and *HQ* locates one point, *s*, of the tangent line *T*. A second point is the given point 10 of the intersection, which is here used instead of finding the *V*-trace, *VS*, of the tangent plane.

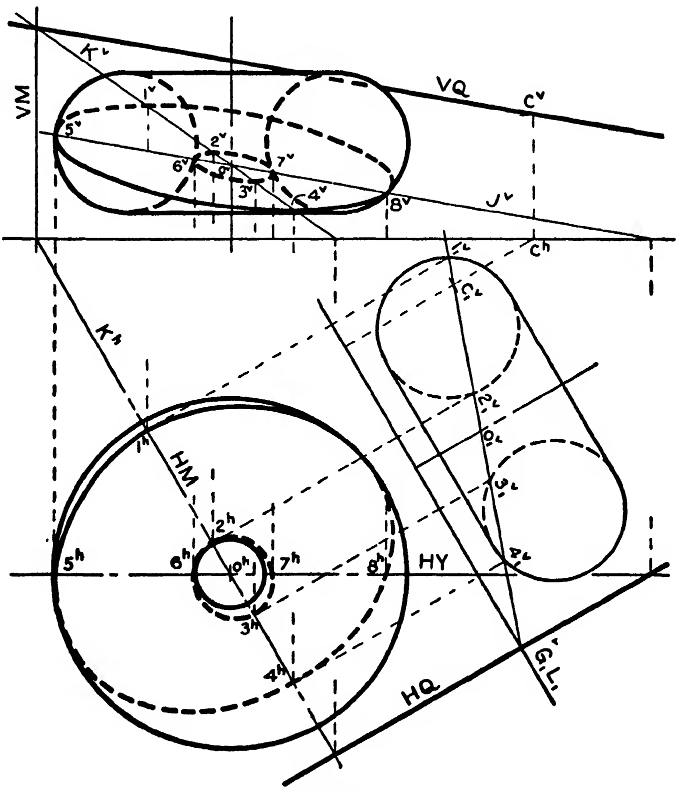

Fig. 313.

183. Points in Meridian Planes. Attention has already been called, in the preceding constructions, to certain properties of

the points which lie in the principal meridian plane and the meridian plane of symmetry. These points are of sufficient importance to warrant a further discussion. In an actual construction, instead of finding these points last, as was done in explaining the preceding problem, it is better to find them at the outset. This is shown in Fig. 313, in which are found

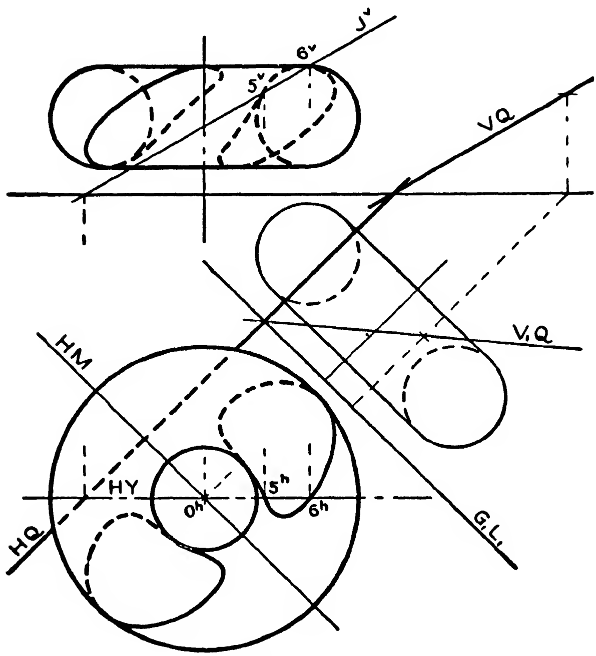

FIG. 314.

only the points lying in the meridian plane of symmetry *M*, and the principal meridian plane *Y*, the construction being the same as that used in Fig. 312. The construction is repeated in Fig. 314 for the same planes *M* and *Y*. In this case we find that there are no points in the meridian plane of symmetry *M*.

184. The Number of Curves. The intersection of a plane and a torus always consists of either one or two closed curves. The number can be foretold as soon as the edge view of the secant plane is obtained. Thus, in Fig. 312, the edge view

V_1Q of the secant plane intersects but one of the circles lying in the meridian plane of symmetry M. By visualizing the position of the cutting plane with respect to the torus, it can be seen that the intersection will consist of but one curve, with the point 11 as its lowest point.

In Fig. 313, the edge view V_1Q intersects both circles lying in the meridian plane of symmetry M, giving four points, 1,

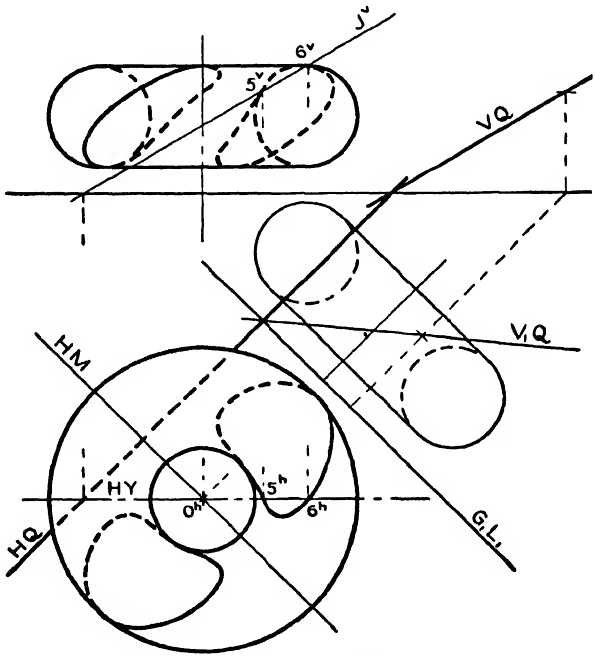

Fig. 314 (repeated).

2, 3, and 4. Here, again by visualization, the intersection is seen to consist of two distinct curves, one wholly inside the other. The point 1 is the highest, and the point 4 the lowest point of the intersection; no auxiliary planes need be taken beyond these limits. It may be noted in passing that these four points lie on the line of intersection, K, of the planes M and Q, and that their V-projections are here determined as lying on K^v. (Compare Fig. 311.)

In Fig. 314, the edge view V_1Q does not intersect either circle lying in the meridian plane of symmetry *M*, and, as previously noted, there are no points in this plane. Yet the plane *Q* obviously intersects the torus. Visualization here reveals the fact that the intersection in this case consists of two separate curves, symmetrically placed with respect to the plane *M*.

It occasionally happens that the edge view of the given secant plane is found to be tangent to one or both of the circles lying in the meridian plane of symmetry. In this event, the plane is actually tangent to the torus, yet at the same time intersects the surface in a curve.

A point of tangency of the secant plane gives a double point in the curve of intersection; that is, a point in which

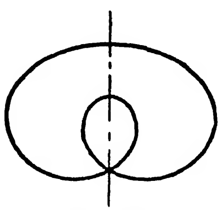

FIG. 315.

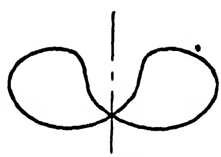

FIG. 316.

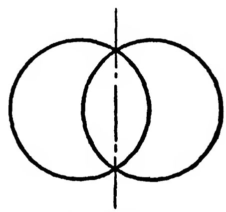

FIG. 317.

the curve crosses itself at an angle. According to whether the plane does or does not intersect the second circle lying in the meridian plane of symmetry, the resulting projection resembles Fig. 315 or 316, respectively. Each of these forms is considered a single curve.

If the given secant plane is tangent to the torus at two points, there are two double points in the intersection (Fig. 317). In this case it can be proved analytically that the actual intersection consists of two equal circles.

Summarizing, it may be stated that if the edge view of the secant plane is found in the same relation to both meridian circles of the torus, that is, secant to both, or tangent to both, or passing between the two without touching either, there will always be two curves. Any other position of the secant plane means one curve.

CHAPTER XXIV

THE INTERSECTION OF CURVED SURFACES BY CURVED SURFACES

185. The Intersection of Two Curved Surfaces. The intersection of two curved surfaces is, in general, a space curve, or curve of three dimensions (§ 150). Points in this curve may be found as follows. Let S_1 and S_2 be the intersecting surfaces. Let X be any auxiliary surface, intersecting S_1 in a line C_1, and S_2 in a line C_2. Then the point or points of intersection of C_1 and C_2 lie in the line of intersection of S_1 and S_2. In order that the application of this method shall be a success, it is evident that the intersections of X with S_1 and S_2 must be simpler or more easily found than the intersection of the given surfaces S_1 and S_2.

186. The Auxiliary Surfaces. Whenever possible, the auxiliary surfaces employed are planes, so chosen as to intersect both the given surfaces in either straight lines or circles. This cannot always be accomplished, even when straight lines or circles can be drawn on each of the given surfaces.

Spheres are employed occasionally as auxiliary surfaces. Their use is generally limited to cases in which they can be so placed as to intersect each of the given surfaces in circles.

When no auxiliary surfaces can be found which will cut both given surfaces simultaneously in simple intersections, it is usual to employ planes. These are placed as advantageously as possible with respect to one of the given surfaces, but their intersections with the other given surface must needs be found by means of secondary auxiliary surfaces.

187. Visibility of the Curve of Intersection. When two surfaces or solids intersect, both will be considered opaque, and no portion of either removed unless so stated. (Compare

§§ 85, 125 *et seq.*) In any projection, then, a point in the curve of intersection can be visible only when it lies on a visible portion of each of the given surfaces. Or, to put it another way, a point is invisible if it is on an invisible portion of either of the given surfaces.

In any projection, let the visible portion of the curve of intersection be considered as traced by a moving point. Unless the entire intersection is visible, which is possible but rarely happens, the moving point will soon reach a place where it can go no further on the curve, an invisible portion of the curve having been reached. This change must evidently take place on the contour or outline of the surface. Hence from this point on, the contour of the surface must be visible, and may be considered as the continuation of the path of the moving point. This is a matter often overlooked by students in determining the combined visibility of a pair of intersecting solids.

188. A Line Tangent to the Curve of Intersection. In general, a line tangent at any point in the intersection of two curved surfaces may be determined by finding the line of intersection of two planes. For the chosen point lies at the same time in each of the given surfaces, to each of which the line tangent to the curve is tangent. Then a plane passed tangent to either surface at the given point must contain the line in question, since a tangent plane contains all the lines tangent to the surface at that point (§ 154). Hence we may state the following rule:

1. *Pass a plane tangent to each of the given surfaces at the given point.*

2. *Find the line of intersection of these two planes. This line is the required tangent line.*

It will be observed that this rule is perfectly general, and applies to the intersection of any two curved surfaces whatever. The rule fails when the two tangent planes coincide, which occurs at a point where there is a multiple or isolated point in the intersection.

CHAPTER XXV

THE INTERSECTION OF CONES AND CYLINDERS WITH EACH OTHER

189. The Intersection of Cones and Cylinders. The intersection of two surfaces, each of which may be either a cone or a cylinder, gives rise to three problems; viz. the intersection of

(a) *Two cylinders.*

(b) *Two cones.*

(c) *A cylinder and a cone.*

These are essentially three cases of a single problem; namely, the intersection of two single curved surfaces, and the same general principles can be applied throughout. In considering the details of construction, however, it is more convenient to treat the cases as separate problems.

190. The Auxiliary Planes. In finding the intersections of cylinders and cones, we shall employ planes as the auxiliary

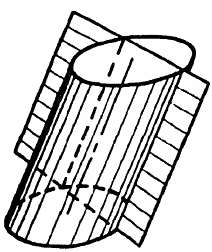

FIG. 318.

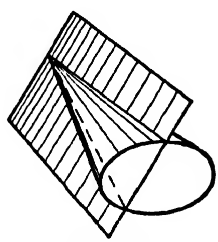

FIG. 319.

surfaces (§ 185). It is always possible to place planes so that they will cut elements, that is, straight lines, from each of the given surfaces. To verify this statement, let us study the conditions which are thus imposed upon the planes.

To cut elements from a cylinder, the secant plane must be parallel to the axis of the cylinder (Fig. 318). This is a

necessary, but not a sufficient condition, since a plane may evidently be parallel to the axis of a cylinder, and yet so far removed as not to intersect at all. But if the plane does contain one or two elements, it is parallel to the axis. This is the condition imposed by the cylinder.

A plane which contains elements of a cone must contain the vertex of the cone (Fig. 319). Hence the cone imposes the condition that all auxiliary planes must contain its vertex.

Considering now the three cases, the auxiliary planes must fulfill simultaneously the following conditions.

(*a*) *Two cylinders.* The planes must be parallel to the axis of each cylinder.

(*b*) *Two cones.* The planes must contain the vertex of each cone.

(*c*) *A cylinder and a cone.* The planes must contain the vertex of the cone, and be parallel to the axis of the cylinder.

191. The Number of Curves. We confine ourselves to cones and cylinders of the ordinary closed forms, with circles or ellipses for bases (§ 158). In the general case of intersection of two such surfaces, where no element of either surface is parallel to any element of the other surface, the intersection will consist either of one or of two closed curves (curves of three dimensions, § 185). When each solid cuts into, but not wholly through, the other solid, there will be a single curve of intersection. But if every element of either solid actually intersects the other solid, there will be two separate curves. In either case, the projections of these curves on the coördinate planes may, and generally do, contain cusps and loops, although there are none in the curves themselves.

Special cases, resulting from parallel elements, or particular positions of the solids, will be better understood after some actual intersections have been found. The discussion of these cases is therefore deferred. (See § 193 *et seq.*)

192. General Cases. We shall now take up the general cases, in which the intersection consists of either one or two closed curves.

Problem 46. *To find the intersection of two cylinders.*

Analysis. Assume any point in space. Through this point pass the plane which is parallel to the axis of each cylinder (Prob. 8, § 107). An auxiliary plane parallel to this plane will intersect each cylinder in elements of the surface (§ 190). The points in which these elements intersect will lie in the curve or curves of intersection of the cylinders (§ 185). Evidently the auxiliary planes must be so chosen that they will contain at least one element of each surface.

EXAMPLE 1. **Construction** (Fig. 320). The given cylinders lie in the first quadrant, with their bases on *H*. They are distinguished by the letters *A* and *B*. Let o (o^h, o^v) be any assumed point in space. Through o draw the line *K*, parallel to the axis of the cylinder *A*, and the line *L*, parallel to the axis of the cylinder *B*. Find the plane *X* which contains these two lines. Only the *H*-trace, *HX*, is here found, as it will soon be seen that in this case it is sufficient. Planes parallel to *X* will cut elements from each cylinder. Let *HM* be the *H*-trace of a parallel plane. By noting where *HM* intersects the base of each cylinder, we see that *M* cuts the elements *C* and *D* from cylinder *A*, and the elements *E* and *F* from cylinder *B*. The intersections of these elements determine four points, 3, 7, 11, and 15, on the required curve.

A. CHOOSING THE AUXILIARY PLANES. THE LIMITING PLANES. Planes parallel to *M* and behind it will intersect each cylinder in two elements until we reach the plane *Y*, which is tangent to the cylinder *B*. This plane contains two elements of *A*, but only one of *B*, which intersect in but two points, 1 and 9, of the curve. Beyond *Y* a parallel plane would cease to intersect *B*, hence we have reached the limit of useful auxiliary planes in this direction. (See the Analysis.)

Coming forward from the plane *M*, we reach the limit of useful planes with the plane *Z*, tangent to the cylinder *A*. A plane in front of *Z* will contain no elements of the cylinder *A*.

The planes *Y* and *Z* are called the **limiting planes.** They are the first auxiliary planes which should be drawn. As soon as they are located, the number of intersections — one or two

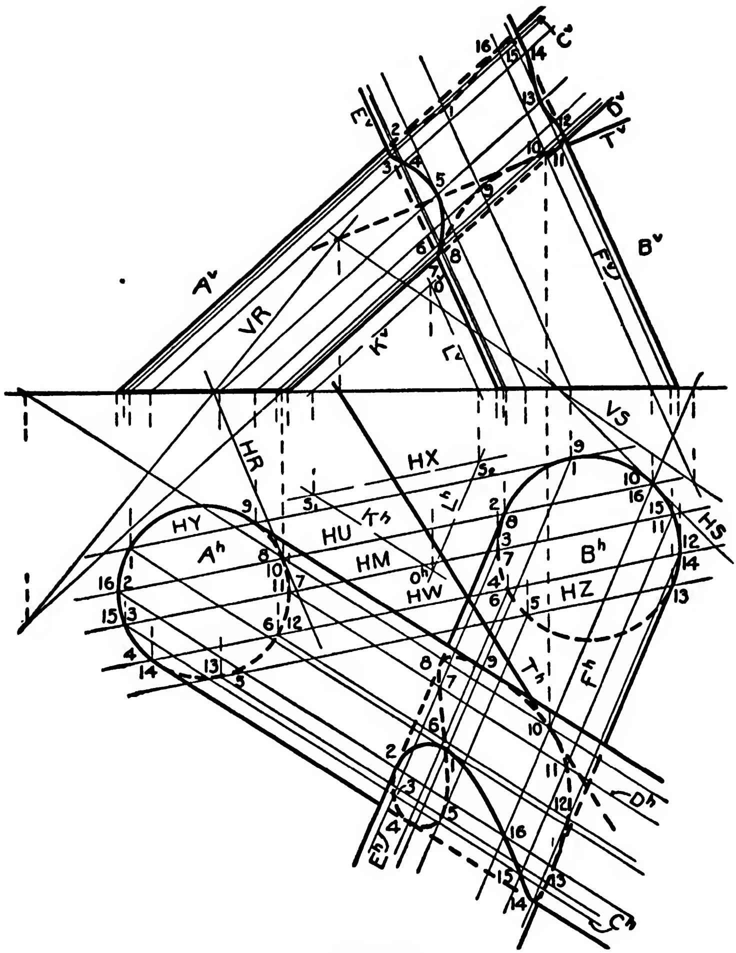
C^v
D^v
T^v
E^v
A^v
VR
K^v
L^v
F^v
B^v
VS
HR
HX
HY
HU
K^h
L^h
A^h
HM
O^h
HW
B^h
HZ
HS
T^h
F^h
D^h
E^h
C^h

FIG. 320.

(§ 191) — can be foretold. In the present case, one limiting plane is tangent to one of the solids, the other plane tangent to the other solid. This means that each solid cuts part way, but not completely through the other, and the intersection consists of one continuous curve.

After the limiting planes are drawn, a sufficient number of intermediate auxiliary planes are chosen. In choosing these planes, they should be passed, as far as possible, so as to contain the contour (outside) elements of the various projections. Thus the plane U is passed through the point 2–16 in the base of the cylinder A, and contains the upper contour element of the V-projection of this cylinder. The plane W passes through 4–14 on the base of A, and contains the front contour element of the H-projection. Incidentally, these same planes also contain contour elements of the cylinder B, which, however, cannot always be expected to happen.

B. Numbering the Points of the Intersection. To locate the projections of the points, and to connect these projections in the proper sequence after they are obtained, a system for numbering the points as fast as found is desirable. Take some point of the curve of intersection as the initial point, number 1. This point is preferably one of the points in one of the limiting planes. Here the point taken lies in the plane Y. Observe the H-projection of this point. Follow the two elements, one of each cylinder, passing through this point, to the respective bases, and place duplicate numbers 1 on HY as shown.

Point 2 must lie in the plane adjacent to Y, that is, in plane U, and also on the element in each cylinder adjacent to that numbered 1. On the base of A there is no choice, and point 2 is placed on HU as shown. On the cylinder B either element lying in the plane U is adjacent to element 1. Choice is here made as shown by the number 2 on the base of B. That is, having chosen our starting point on the curve, we may also choose the direction in which we shall go around.

For point 3 and subsequent points there is no choice. Point 3 must lie in the plane M, and on the bases of both A and B

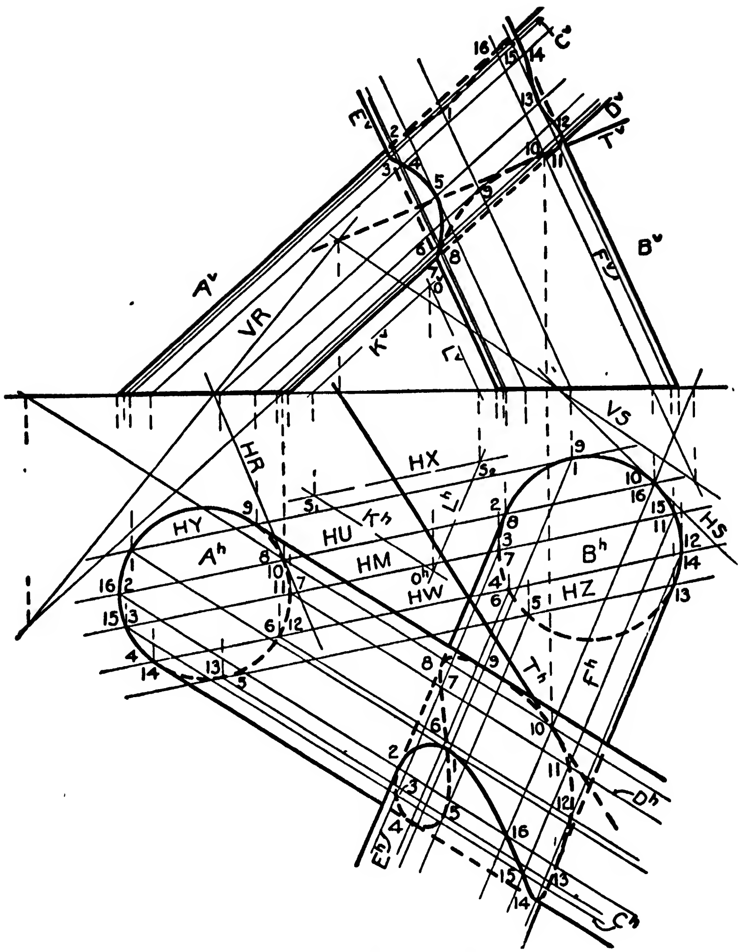

FIG. 320 (repeated).

the number 3 must be placed on the nearer element to 2. Continuing, the two 4's are put on HW, and the two 5's on HZ. We have now followed the curve across all the planes; the curve, in continuing, crosses the planes in reverse order, so that point 6 is in the plane W, point 7 in plane M, and so on. To place the two 6's on HW, note that on the base of B the 6 must be placed on the same element as 4, since this is the nearer element in cylinder B to the element 5. Consequently on the base of cylinder A the number 6 must not be placed on the same element as 4, as that would make point 6 on the curve identical with 4. Hence point 6 on the base of A can be placed only as shown. The two 7's on HM, the two 8's on HU and the two 9's on HY follow naturally. Having again followed the curve across the planes, and not yet reached the starting point, we must again turn back. To place the two 10's on HU, the argument is similar to that for placing the two 6's on HW. The remainder of the numbers follow in order, presenting no new difficulties. Finally, reaching the two 16's on HU, we see that the next point on each base is the initial point 1, and we have completed the circuit.

In an actual construction, the duplicate numbers on the bases should be placed before any elements are drawn. There is a noticeable symmetry in their placing, provided we start with a point in a limiting plane, which enables this to be done. The elements are then drawn, and the points in the intersection are marked with the same numbers. The points may then be readily connected in order.

C. Visibility of the Curve of Intersection. A point in the intersection is visible only when each of the elements, one of each surface, in which it lies, is visible (§ 187). The visibility in any projection is most readily determined by constructing a table of the visible points, as shown below. The points on contour elements are to be included as visible.

D. Visibility in the H–Projection.

Cylinder A, if alone — Points 1, 2, 3, 4, 9, 14, 15, 16.
Cylinder B, if alone — Points 1, 2, 8, 9, 10, 11, 12, 13, 14, 15, 16.

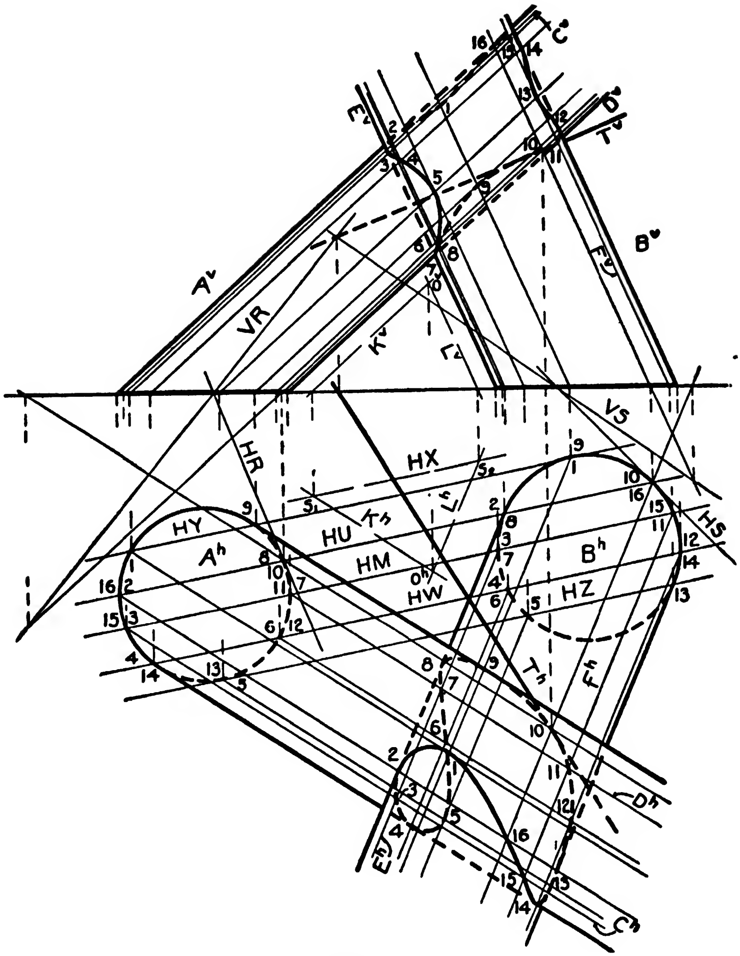

FIG. 320 (repeated).

The Intersection, being the points common to the two preceding lists — Points 1, 2, 9, 14, 15, 16.

Line in the portions 1–2 and 14–15–16 of the H-projection of the curve. Since the numbers on the curve form an endless circuit, point 1 follows 16, and the portion 16–1 is visible. The isolated point 9 being found visible, it means that the entire contour element passing through 9 is visible. The other visible contour elements, tangent from 2 and 14, are now put in by the rule given in § 187.

E. VISIBILITY IN THE *V*-PROJECTION.

Cylinder A. Points 2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16.

Cylinder B. Points 3, 4, 5, 6, 7, 12, 13, 14.

The Intersection. Points 3, 4, 5, 6, 7, 12, 13, 14.

In the V-projection, then, there are two visible portions of the curve, 3–4–5–6–7 and 12–13–14. At point 7 the projection forms a cusp, and is not tangent to either contour element. The point 7 in fact lies on each of these contour elements, and the tangent at this point to the curve in space is perpendicular to V. The visible contour elements are those tangent from points 3, 12, and 14.

F. A LINE TANGENT TO THE CURVE OF INTERSECTION.

Analysis. See § 188.

Construction. In order that the general construction may be shown, it is necessary to select a point which does not lie in a limiting plane, nor on a contour element. At point 1, for example, the curve is tangent to the element 1 of cylinder A, while at a point 2 the projection of the tangent in each view coincides with a contour element of one of the cylinders.

Let point 10 be chosen. Pass the plane R, tangent to the cylinder A at point 10 (Prob. 34, § 162); also the plane S, tangent to the cylinder B at point 10. The line of intersection, (T^h, T^v), of these two planes is the required tangent line. In this case, the intersection of HR and HS is inaccessible. However, no auxiliary construction is necessary, since the projections of T can be readily determined by the intersection of VR and VS, and the fact that T must pass through point 10 of the curve.

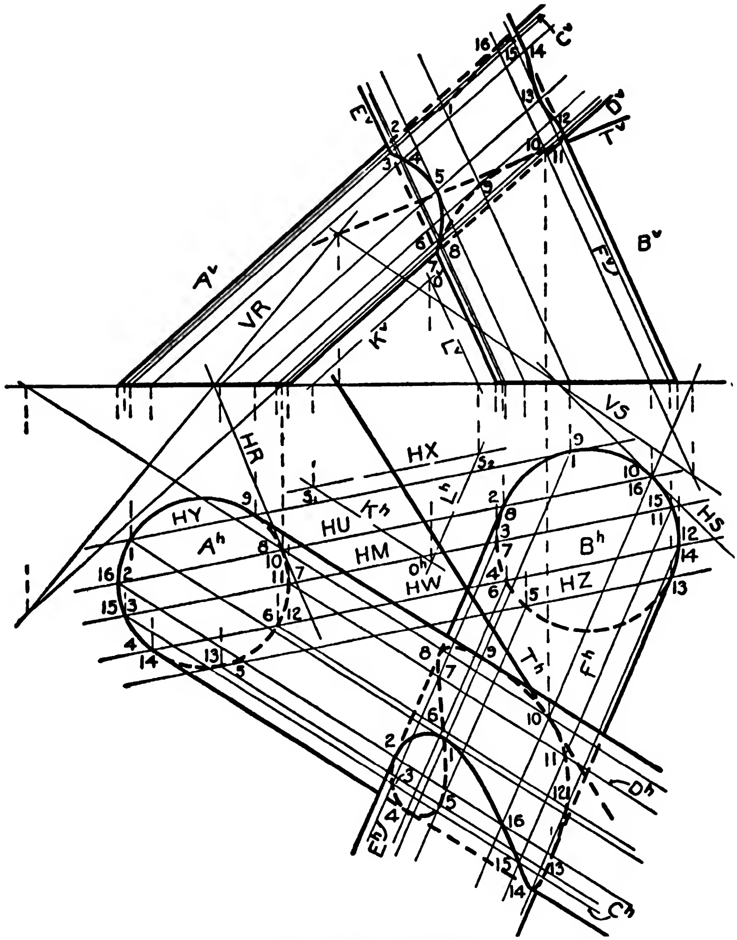

FIG. 320 (repeated).

EXAMPLE 2. **Construction** (Fig. 321). The base of the given cylinder *A* lies in *H*, while the base of *B* lies in *V*. This will necessitate the use of both traces of the auxiliary planes.

Let *o* (o^h, o^v) be any assumed point in space. Through *o* draw line *K*, parallel to the elements of cylinder *A*, and line

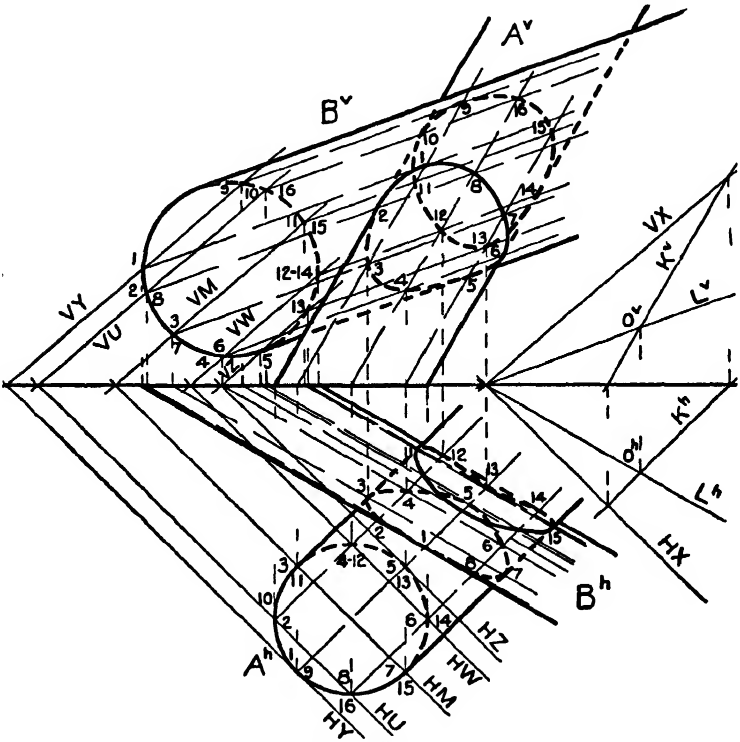

FIG. 321.

L, parallel to the elements of cylinder *B*. Find the plane *X* (*HX*, *VX*) passing through these lines. All the auxiliary planes must be parallel to *X*.

The limiting planes *Y* and *Z* are drawn first. The fact that both these planes are tangent to the base of *A* can only be found by trial. Planes whose *V*-traces are tangent to the base of *B* can be drawn parallel to the plane *X*; but if drawn, it will be found that the *H*-traces of these planes do not intersect

the base of A. These planes must therefore be rejected, since they contain no element of the cylinder A (§ 190).

Both limiting planes being found tangent to the same solid, that is, in this case cylinder A, it means that A completely penetrates the other cylinder, every element of A intersecting cylinder B. Hence there are two separate curves of intersection.

The auxiliary planes between the limits Y and Z are chosen so as to contain the various contour elements of the surfaces, as in the previous example.

A. Numbering the Points. The method of numbering the points for a two-curve intersection differs in detail, but not in principle, from the method of the one-curve intersection of Example 1.

As every element of A intersects B, it follows that each curve passes completely around cylinder A, but only part way around cylinder B. Consider each curve separately. Then, placing the duplicate numbers 1 on one of the limiting planes, as Y, the numbers on the base of A will follow consecutively in the same direction entirely around the base, while for the base of cylinder B, the numbers will proceed as far as 5, then reverse direction as shown. Commencing with 9 for the second curve, the base of A is numbered completely around in one continuous direction (either the same direction as before, or the opposite), while on the base of B the numbers run from 9 to 13, and then reverse.

The projections of the points in the intersection are now obtained readily, and connected in the order of the numbers.

B. Visibility of the Curves of Intersection. The visibility is determined, as in Example 1, by constructing tables of the visible points.

C. Visibility in the H-projection.

Cylinder A. Points 1, 2, 3, 7, 8, 9, 10, 11, 15, 16.

Cylinder B. Points 1, 9, 10, 11, 12, 14, 15, 16.

Intersection. Points 1, 9, 10, 11, 15, 16.

The result is the isolated point 1 on one curve, and the continuous line 15–16–9–10–11 on the other.

D. Visibility in the *V*-projection.

Cylinder A. Points 1, 2, 6, 7, 8, 9, 10, 14, 15, 16.

Cylinder B. Points 1, 2, 3, 4, 5, 6, 7, 8.

Intersection. Points 1, 2, 6, 7, 8.

The result is the continuous portion 6–7–8–1–2 of one curve. The other curve is entirely invisible.

After the visible portions of the curves of intersection have been lined in, the visible contour elements are determined as explained in Example 1.

Problem 47. ***To find the intersection of a cylinder and a cone.***

Analysis. Through the vertex of the cone draw a line parallel to the elements of the cylinder. Auxiliary planes passed through this line will intersect each of the given surfaces in elements (§ 190). The intersection is then determined as in the case of the two cylinders (Prob. 46).

Construction (Fig. 322). The cone A is to be intersected by the cylinder B. Through the vertex a (a^h, a^v) of the cone, draw the line L (L^h, L^v) parallel to the elements of the cylinder. Since the base of one solid lies in H, while the base of the other is in V, both traces of the auxiliary planes will be needed (Prob. 46, Ex. 2). Hence find both the H-trace, s, and the V-trace, t, of the line L. Then all the H-traces of the planes will pass through s, and all the V-traces through t (§ 98). The limiting plane Y is found to be tangent to the cone, while the limiting plane Z is tangent to the cylinder. This means that the intersection will consist of one continuous curve (Prob. 46, Ex. 1). The remainder of the construction involves nothing that has not been explained in the preceding problem.

A Line Tangent to the Curve of Intersection. The line T (T^h, T^v) is tangent to the intersection at point 12. To find this line, pass the plane R, tangent to the cone, along element 12 (Prob. 31, § 162), and the plane S, tangent to the cylinder, along element 12 (Prob. 34, § 162). Then find the line of intersection of these two tangent planes (§ 188; Prob. 46, Ex. 1).

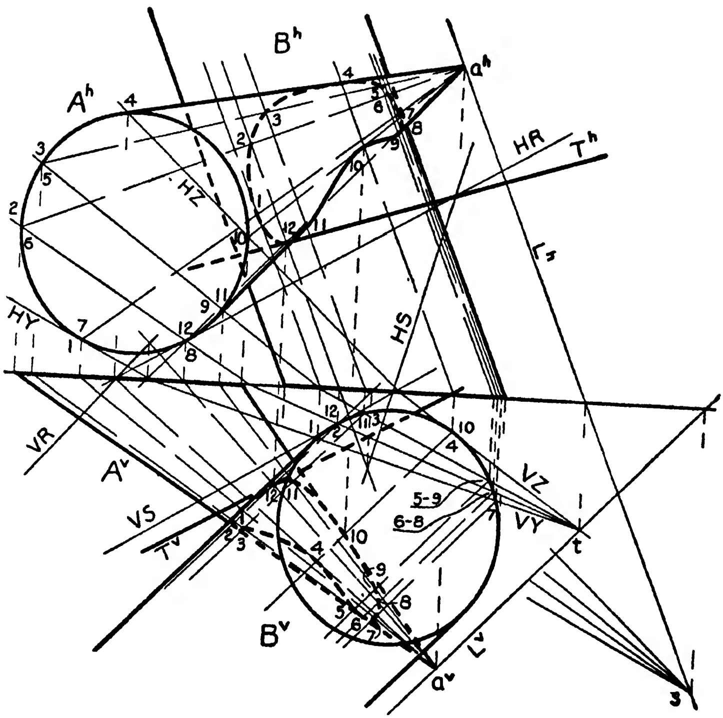
B^h
a^h
A^h
4
3
5
2
6
7
8
9
10
11
12
HZ
HR
T^h
L^h
HY
HS
VR
A^v
VS
T^v
VZ
VY
5-9
6-8
t
B^v
a^v
L^v
s

Fig. 322.

Problem 48. *To find the intersection of two cones.*

Analysis. Draw the line which passes through the two vertices of the cones. Auxiliary planes passed through this line will intersect each of the given surfaces in elements (§ 190). The intersection is then determined as in the case of two intersecting cylinders (Prob. 46).

Construction (Fig. 323). The given cones are A and B. Draw the line L (L^h, L^v) which connects the vertices, a and b, of the two cones. Find the H-trace, s, of this line: then all the H-traces of the auxiliary planes pass through s (§ 98). Since the bases of both cones lie in H, only the H-traces of the auxiliary planes are needed (Prob. 46, Ex. 1). The limiting planes Y and Z are both tangent to the base of A, showing that the intersection consists of two separate curves (Prob. 46, Ex. 2). The remainder of the construction involves nothing that has not been already explained.

The line T is tangent to the intersection at point 10. It is the line of intersection of the plane R, tangent to the cone A along the element a–10, and plane S, tangent to cone B along the element b–10 (§ 188).

193. Special Cases. There are two general reasons for special cases of the preceding three problems: (*a*) *limiting planes tangent to both surfaces simultaneously;* (*b*) *parallel elements.*

194. Limiting Planes Doubly Tangent. If a limiting plane is tangent to both surfaces, it contains but a single element of each surface. These elements intersect in a point through which both branches of the curve of intersection must pass. The intersection is a continuous curve, containing one double point.

If both limiting planes are tangent to each surface, the intersection breaks up into two plane curves. Since the only plane curves which can be drawn on a cylinder are circles and ellipses, the intersection will take these forms when one or both of the given surfaces is a cylinder. If both surfaces are cones, the intersection may consist of any of the conic sections.

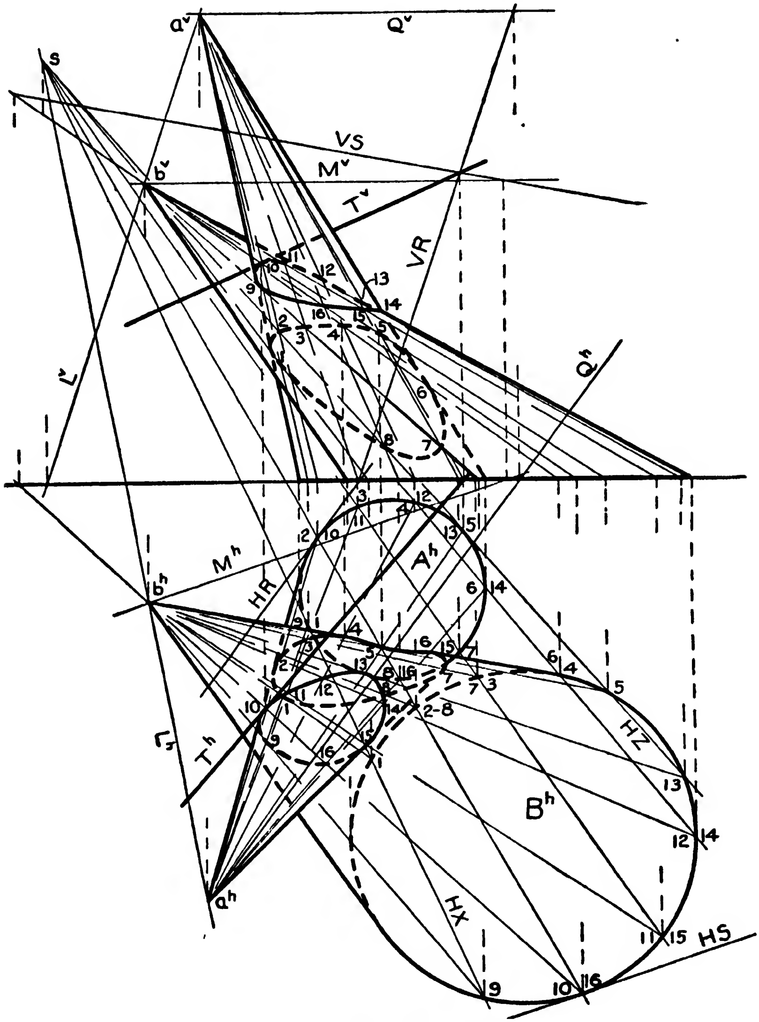

FIG. 323.

Thus, in Fig. 324, which shows the *V*-projection only, the plane *X*, parallel to the left-hand contour element of the cone whose vertex is *a*, cuts a parabola from this cone. A cone

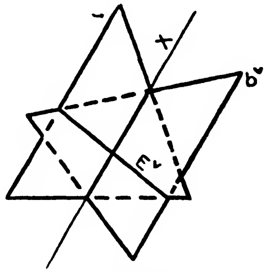

FIG. 324.

whose vertex is *b* can evidently be placed so as to contain the same parabola, which thus becomes the line of intersection of the two cones. In this case the cones intersect again in a closed curve, an ellipse or circle *E*.

If two cones are identical in form, and placed so that their corresponding elements are parallel (Fig. 325), the intersec-

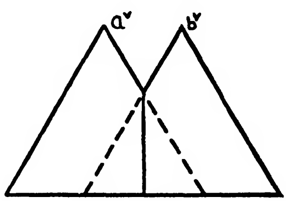

FIG. 325.

tion is one branch of a hyperbola. In this case there is no other intersection on the two nappes of the cone which are here shown; but the upper nappes of the cones will intersect in the second branch of the same hyperbola.

195. Parallel Elements. If two elements, one in each surface, are parallel, the intersection of the surfaces may, though not necessarily, have one or more infinite branches.

A. Two Cylinders. In the case of two cylinders, if there is one pair of parallel elements, all the elements of both surfaces are parallel, and the intersection can consist only of straight lines.

B. A Cylinder and a Cone. In the case of a cylinder and a cone, since no two elements of the cone are parallel, there can be but one element of the cone parallel to the elements of the cylinder; but this element of the cone will then be parallel to all the elements of the cylinder. Hence, when we start to find the intersection by drawing a line through the vertex of the cone parallel to the elements of the cylinder, this line coincides with an element of the cone.

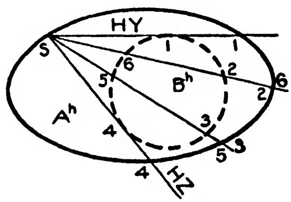

Fig. 326.

In Fig. 326, which shows an *H*-projection only, let *A* be the base of the cone, and *B* the base of the cylinder, both bases in *H*. Let the *H*-trace of the line through the vertex of the cone parallel to the elements of the cylinder be at *s*, which is necessarily on the base of *A*, for reasons which have been given. In the situation shown, since both limiting planes *Y* and *Z* are tangent to the same base *B*, we might expect a two-curve intersection. One intersection is not affected by the parallel elements, and is a closed curve, found by the elements 1 to 6. The second intersection does not exist, since the single element through point *s* does not intersect the cylinder. Hence the intersection consists of a single closed curve.

Now let the cone and cylinder be placed as in Fig. 327, which differs from Fig. 326 in that the plane *M*, tangent to the cone along the parallel element, has now become one of the auxiliary planes. The single intersection of the preceding

case now contains two infinite points, numbers 3 and 7. The result is that the curve of intersection, as traced on the surface of the cylinder, consists of two infinite branches, to which the elements 3 and 7 are asymptotic. These two branches of the intersection lie on opposite nappes of the cone.

The preceding two examples do not exhaust all the possibilities of this case. They are, however, the most general ones, and, taken together, give a typical illustration of the formation of infinite branches.

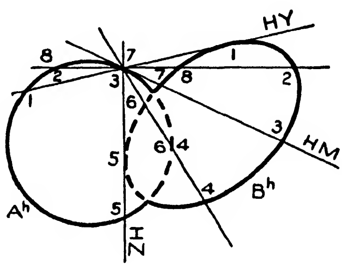

Fig. 327.

C. Two Cones. In the case of two cones, to discover the number of parallel elements, draw a third cone, whose vertex coincides with the vertex of one of the given cones, and whose elements are parallel respectively to the elements of the other. Then, corresponding to the number of possible points which two ellipses may have in common, we may find one, two, three, or four pairs of parallel elements, or every element of one cone may be parallel to an element of the other cone. The latter case is that already illustrated in Fig. 325, the intersection being a hyperbola. The possibilities of the other cases are quite numerous. As an extreme result, each curve of a normal two-curve intersection may be broken into two infinite parts, so that the complete intersection consists of four infinite branches, two branches on each nappe of each cone. Since the method of dealing with infinite points has already been shown for the case of the cylinder and cone, a further discussion will not be made.

196. The Intersection of Cylinders and Cones of Revolution. If we limit ourselves to cylinders and cones of revolution, usually the case in practice, simpler solutions than those given in § 192 can often be found. It may be of advantage to take auxiliary planes so as to intersect one or both of the given surfaces in circles. For example, consider the intersection of a cylinder and cone, the axes of which are at right angles (not necessarily intersecting). Auxiliary planes can be passed perpendicular to the axis of the cone, cutting the cone in circles, and the cylinder in straight lines (elements).

If the axes of the surfaces intersect at some oblique angle, auxiliary spheres may be used. (See Prob. 49, Ex. 2, § 197.)

A method available when one, at least, of the given surfaces is a cylinder, is to make a projection on a plane at right angles to the axis of the cylinder. The entire curved surface of the cylinder will then project as a circle, a part or all of which necessarily becomes one projection of the required intersection of the given surfaces.

CHAPTER XXVI

THE INTERSECTION OF VARIOUS CURVED SURFACES

197. The Intersection of Two Curved Surfaces of Revolution. The intersection of two surfaces of revolution can be readily found when the axes of the surfaces are in the same plane, that is, either intersect or are parallel.

Problem 49. ***To find the intersection of two surfaces of revolution whose axes intersect.***

Analysis. With the point in which the axes of the surfaces intersect as center, describe a series of auxiliary spheres (§ 186). These spheres will intersect both given surfaces in circles. The intersections of the corresponding circles locate points in the required intersection.

In order to get simple working projections of the auxiliary circles, one coördinate plane should be parallel to the two axes of the surfaces, the other coördinate plane perpendicular to one of the axes. (See the Construction.)

EXAMPLE 1. **Construction** (Fig. 328). The given surfaces are those of a cone and an ellipsoid of revolution, placed in the third quadrant. The plane Y (HY) which contains the axes of both surfaces is parallel to V. This plane is evidently a plane of symmetry of the required intersection, and contains two points, noted directly in the V-projection at 1^v and 2^v.

The axes of the given surfaces intersect in the point o (o^h, o^v). With o^v as center, and any assumed radius, draw the outline, S^v, of an auxiliary sphere. This sphere intersects the cone in two circles, A and B, and the ellipsoid in the circle C, all projected in V as straight lines. The intersections of A^v and B^v with C^v determine the V-projections of four points, 3, 4, 5, and 6, of the required intersection. To obtain the H-projec-

tions of these points, note that the circle C projects as a circle, C^h, in H. Find by projection 3^h, 4^h, 5^h, and 6^h in C^h.

Other points are similarly obtained by the use of other

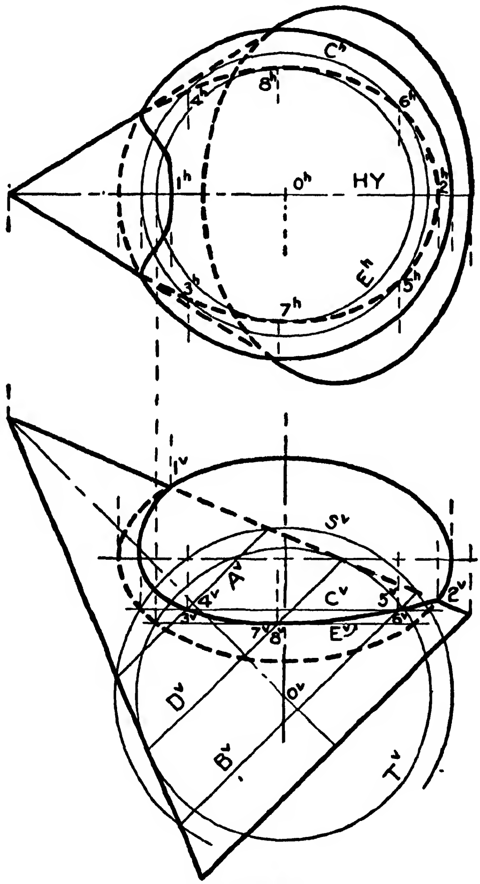

FIG. 328.

auxiliary spheres. The only other sphere shown in the figure is T (T^v), tangent to the cone. This sphere is tangent to the cone in the circle D, and intersects the ellipsoid in the circle E. The intersections of D and E are the points 7 and 8, the lowest points of the curve of intersection.

EXAMPLE 2. **Construction** (Fig. 329). The given surfaces are those of a frustum of a cone and a cylinder of revolution, placed in the third quadrant with their axes parallel to V. The common meridian plane Y of the two surfaces contains the two points 1 and 2.

Other points are found by the use of auxiliary spheres whose centers are at the point of intersection o, of the axes. One sphere, S (S^v), is shown in the figure.

This cuts the circle A from the cylinder, and the circle C from the cone. The intersections of these circles are points 3 and 4 on the intersection. Other points are similarly obtained.

COROLLARY. *To find the intersection of two surfaces of revolution whose axes are parallel.*

Analysis. Pass auxiliary planes perpendicular to the axes of the surfaces. These planes will cut circles from each surface. The intersections of these circles determine points in the required intersection.

In order to get simple working projections of the circles, the axes of the surface should be perpendicular to one of the coördinate planes.

No figure for this case is deemed necessary.

198. The Intersection of a Sphere with Another Surface. When one of two intersecting surfaces is a sphere, the auxiliary surfaces commonly employed are planes, since every plane section of a sphere is a circle. Simple projections of these circles, however, result only when the auxiliary planes are parallel to H or V.

Cases of this kind have already been discussed (§§ 129, 131). But planes parallel to H or V may not always cut advantageous sections from the second surface.

In such cases, the planes are chosen with respect to the second surface, and various devices are adopted to avoid projecting the sections of the sphere as ellipses. Two general methods are (*a*) revolution of the auxiliary planes; (*b*) the use of a secondary projection. These methods will be shown in the following two problems.

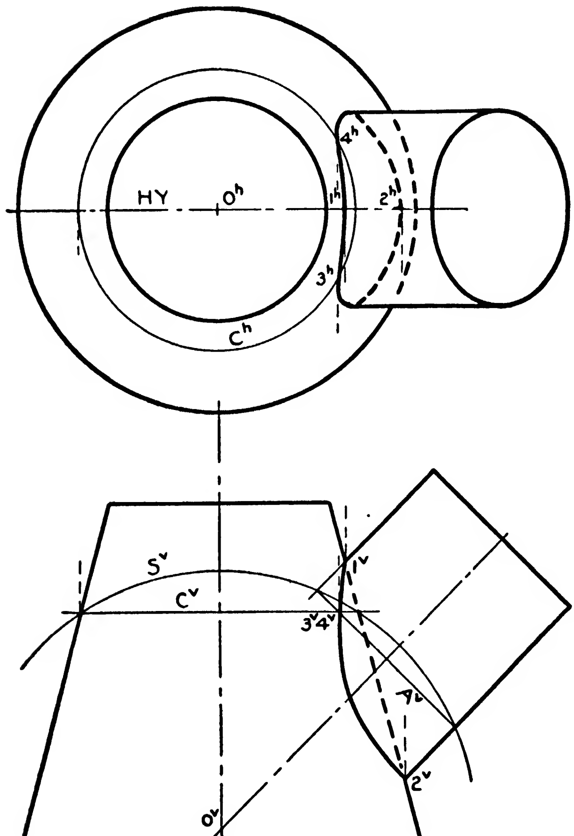

FIG. 329.

Problem 50. *To find the intersection of a sphere and a cone.*

Analysis. Planes passed through the vertex of the cone will cut elements from the cone (§ 190), and circles from the sphere. Let these planes be taken perpendicular to one of the coördinate planes, say to H. Revolve each plane until parallel to V, carrying with it the elements cut from the cone, and the circle cut from the sphere. The circle will now appear in true shape and size. Note the points of intersection of the revolved circle and elements. Obtain their projections by counter-revolution.

A convenient axis about which to revolve the planes is a line perpendicular to H through the vertex of the cone, since all the planes will contain this line.

Construction (Fig. 330). The given surfaces are those of a cone whose vertex is o, and a sphere whose center is e. Let auxiliary planes be passed perpendicular to H through the vertex of the cone. Of these planes, the plane W has been selected as typical, and all the others omitted from the figure for the sake of clearness. The plane W appears in edge view at HW; the trace VW is not drawn, since it is understood that the plane is perpendicular to H. The plane intersects the cone in the elements 0–1 and 0–2. It intersects the sphere in a circle, of which 3^h–4^h is a diameter. The center of this circle projects at 5^h, the middle point of 3^h–4^h. Let the plane W be revolved about an axis passing through o until it coincides with the plane Y, parallel to V. The elements 0–1 and 0–2 will now appear in V-projection as 0^v–1_r and 0^v–2_r. The center, 5, of the circle is found at 5_r. Note that, although we have not the V-projection, 5^v, of point 5, it is not necessary, since point 5 and the center, e, of the sphere are at the same distance above H. With 5_r as center, radius equal to 5^h–3^h or 5^h–4^h, draw the circle C_r, which represents the circle cut from the sphere by the plane W. In this case both elements intersect the circle. This gives four intersections, 6_r, 7_r, 8_r, 9_r, from which the projections of four points in the required intersection are obtained by counter-revolution. Enough other points to determine the intersection are similarly obtained.

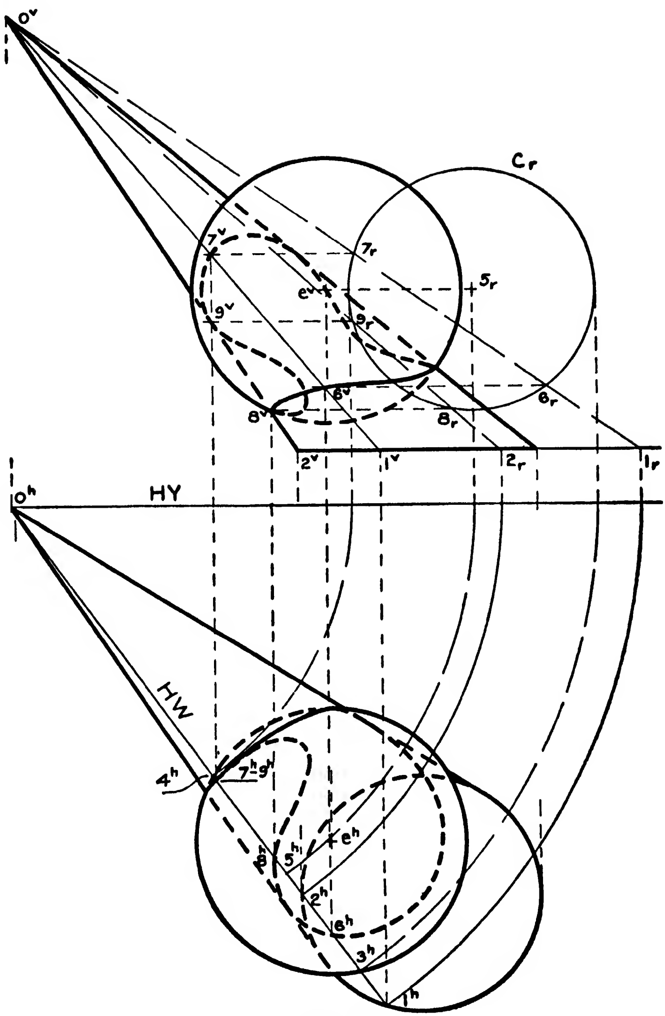
o^v
C_r
7^v
7_r
e^v
5_r
9^v
9_r
6^v
6_r
8^v
8_r
2^v
1^v
2_r
1_r
o^h
HY
HW
4^h
7^h-9^h
e^h
8^h
5^h
2^h
6^h
3^h
1^h

Fig. 330.

Problem 51. *To find the intersection of a sphere and a cylinder.*

Analysis. To cut elements from the cylinder, auxiliary planes must be taken parallel to its axis (§ 190). Let these planes be taken perpendicular to *H*. Assume a secondary plane of projection parallel to the axis of the cylinder. This plane will be parallel to all the auxiliary planes, so that on it all the circles cut from the sphere by the auxiliary planes will appear in true shape and size. Hence project the circle and elements lying in each auxiliary plane to the secondary plane of projection. Note the points of intersection. Project these points back to *H* and *V*.

Construction (Fig. 331). The given surfaces are those of a sphere whose center is o, and a cylinder whose axis is A. Assume a secondary ground line G_1L_1 parallel to the axis of the cylinder. The center of the sphere projects to $o_1{}^v$. To obtain the direction of the elements of the cylinder in the secondary projection, we may project the axis, as shown at $A_1{}^v$. Pass auxiliary planes perpendicular to H parallel to the axis A. The H-trace and edge view of one such plane is shown at HW; all the others are omitted in the figure for the sake of clearness. Plane W intersects the cylinder in two elements, E and F, and the sphere in a circle C. Find the secondary projection of the elements and circle. The elements project as $E_1{}^v$ and $F_1{}^v$, parallel to $A_1{}^v$. The circle projects as $C_1{}^v$, the center coinciding with $o_1{}^v$, while the diameter is obtained from the H-projection. We thus obtain four intersections, $1_1{}^v$, $2_1{}^v$, $3_1{}^v$, $4_1{}^v$. These are the projections of four points in the required intersection from which the H- and V-projections may be found. A sufficient number of points to determine the intersection may be found in a similar manner. In this example the intersection consists of two separate curves.

199. The Intersection of any Two Curved Surfaces. As has been mentioned in § 186, it is not always possible to find a series of auxiliary surfaces which will cut simultaneous simple sections from each of two given surfaces, even when it may be easily possible to cut simple sections from each surface when taken

separately. Such a case, for example, is that of two surfaces of revolution whose axes are not in the same plane, and which, from the nature of the surfaces, does not fall under any of the

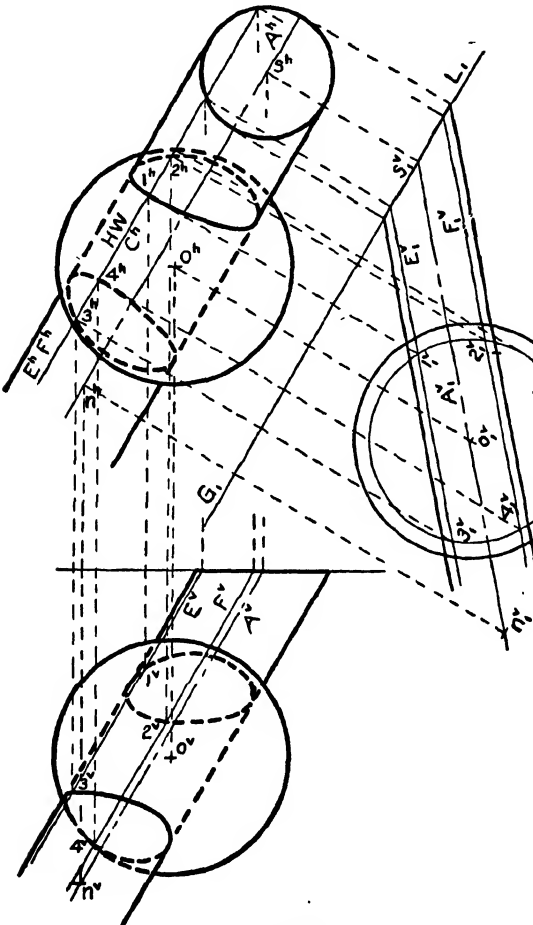

FIG. 331.

cases already discussed. The intersection of two such surfaces, therefore, may well be taken as illustrative of the general method of finding the intersection of any two curved surfaces.

Problem 52. *To find the intersection of any two curved surfaces.* **(General Case.)**

Analysis. It is assumed that neither planes, spheres, nor other auxiliary surfaces can be found which will intersect simultaneously both of the given surfaces in straight lines or circles; or at least that a sufficient number of points to determine the required intersection cannot be found in this way. The solution is then effected by means of auxiliary planes (§ 186). Pass each plane so as to intersect, if possible, one of the given surfaces in straight lines or circles. Find the intersection of this plane with the second surface, using secondary auxiliary planes for the purpose. Note the points of intersection of the two sections; they are points on the required intersection of the given surfaces. In extreme cases, it may be necessary to find each section which the auxiliary plane cuts from the given surfaces by secondary auxiliary planes.

Construction (Fig. 332). The given surfaces are those of a cone of revolution whose vertex is a, and a torus whose center is o. An auxiliary plane M (VM) can be passed through the center of the torus parallel to H, which will cut two circles from the torus, and one circle from the cone. We note in the H-projection the intersection of E^h, the circle lying in the cone, with the circles of the torus, and obtain points 1 and 2 of the required intersection. A plane N (HN) can be passed through the vertex, a, of the cone parallel to V, which will intersect the cone in two straight lines, the contour elements of the V-projection, and the torus in two circles. The V-projection gives us six intersections, thus determining points 3, 4, 5, 6, 7, and 8.

The points already found not being sufficient, and no other simultaneous straight line or circle intersections being possible, recourse must be had to the general method. This is illustrated by the plane R (HR). This plane is taken parallel to V, and intersects the torus in two circles, projected in V at C^v and D^v. Plane R intersects the cone in a hyperbolic arc projected at F^v. The curve F is found by using the secondary auxiliary planes W, X, Y, and Z, parallel to H. (See Fig. 107,

§ 86.) The intersections of F^v with C^v and D^v locate four points, 9, 10, 11, and 12, of the intersection of the surfaces.

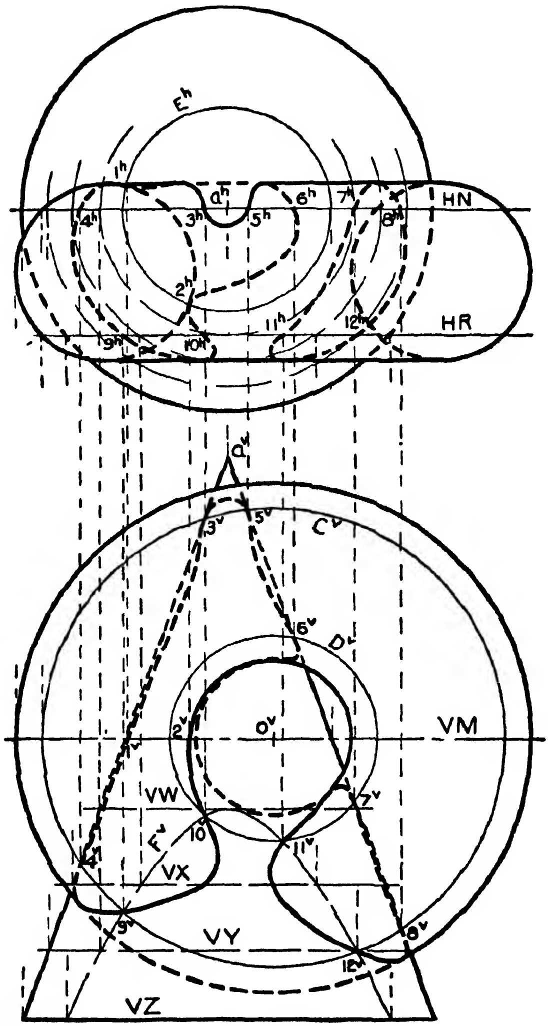

FIG. 332.

Other points to determine this intersection completely may be similarly obtained.

CHAPTER XXVII

PSEUDO-PICTORIAL REPRESENTATION

200. Pseudo-Pictorial Representation. When it is intended to build or construct an object, the various orthographic projections, plan, elevation, side elevation, and so on, cannot always be depended upon to give to the workman a clear idea of the designer's intention. It is often desirable, sometimes practically necessary, to supplement such views by a pictorial representation of the object as a whole, or of some of its parts. Thus, a particular stone in an arch or other masonry structure, a complicated casting in a machine, or an elaborate detail in a building, may need pictorial representation in order to insure against mistakes in construction.

Such a pictorial representation need not be a true picture, or perspective drawing. There have been devised various conventional methods, much more easily constructed than true perspectives, which will serve in such cases just as well as correctly drawn pictures. Such forms of representation may be classed under the head of pseudo-pictorial representation.

201. Orthographic Projection with Pictorial Effect. Under certain conditions, an orthographic projection may have a pictorial effect. At E, Fig. 333, are shown the projections of a square right prism, whose length is equal to twice that of an edge of the base. These projections are constructed by the method of § 72. It will be seen that the vertical projection has a pictorial effect, which is further enhanced if the projection be placed with $o^v c^v$ vertical, as shown at F. (See § 90.)

It is evident, however, that the construction of the elevation at F has involved several processes. In order to be of service as a pictorial representation, a direct method of making this view must be devised.

202. An Axonometric Drawing. The view shown in Fig. 334 has been constructed as follows: Line *OC* is vertical, and equal to the true length of the edge *oc* of the prism of Fig.

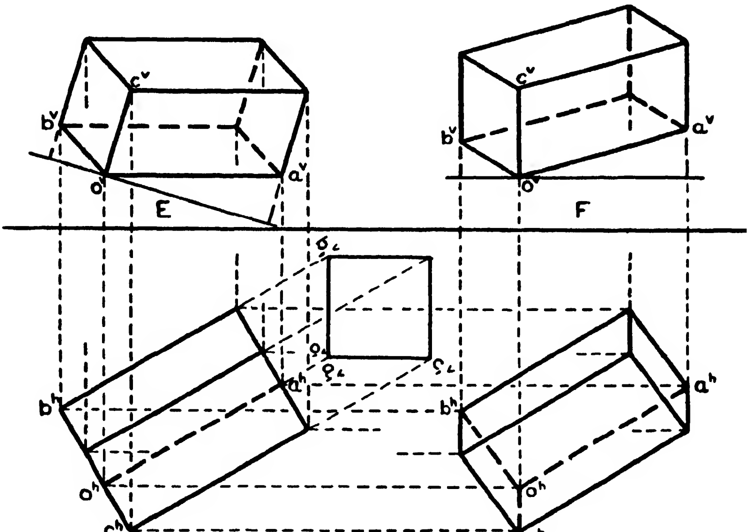

Fig. 333.

333. Line *OA* is inclined at 15° with the horizontal, and equal to the true length of *oa*. Line *OB* is at 30° with the horizontal, and three-fourths of the true length of *ob*. The other lines are parallel to either *OA*, *OB*, or *OC*. The pictorial effect is increased by omitting the invisible lines.

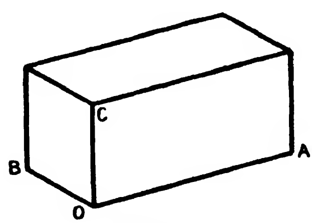

Fig. 334.

Although the drawing of Fig. 334 is somewhat larger than the elevation at *F*, Fig. 333, we have preserved very closely the angles and ratios between the lines which there appear. The result is a satisfactory and easily constructed pictorial representation, known as an axonometric drawing. The three edges of the prism, *oa*, *ob*, and *oc*, mutually perpendicular in space, are

represented by the three lines OA, OB, and OC, called the axonometric axes.

In order that an axonometric drawing may be readily constructed, it is evident that the axonometric axes must be inclined at some angle easily obtainable by the usual draftsman's tools, and that the lengths of these axes must bear simple ratios to the lengths of the same lines in the object.

203. Isometric Drawing. The simplest system of axonometric drawing results when the ratios between the lengths of the axonometric axes and the corresponding edges of a rectangular solid are each made equal to unity. In this case each

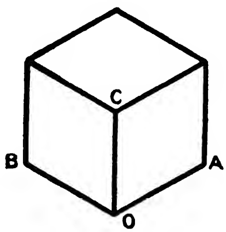

Fig. 335.

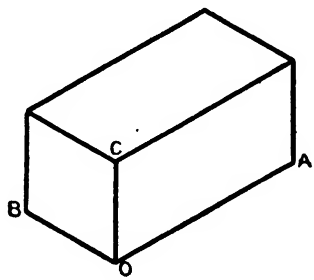

Fig. 336.

axonometric axis must make equal angles, namely 60°, with each of the other two. Such a system of representation is known as isometric drawing (Greek, *isos*, equal, and *metron*, measure).

The isometric drawing of a cube is shown in Fig. 335. The figure shows the axes in the position usually chosen, OC vertical, while OA and OB are each inclined 30° with the horizontal. The lengths $OA = OB = OC =$ the true lengths of the edges of the cube. These lines are the isometric axes.

The isometric drawing of the square prism of Fig. 333 is shown in Fig. 336.

204. Point of View in an Isometric Drawing. An isometric drawing, like any axonometric drawing, is an enlargement of an actual orthographic projection. The plane of this projec-

tion may be found by noting, in the isometric drawing, any line of the object which projects as a point; then any plane perpendicular to this line will serve as the plane of projection. In Fig. 337 is repeated the isometric drawing of the cube of Fig. 335, but with the addition of the invisible edges. It will now be seen that the lower back corner of the cube projects in the same point as the upper front corner. Consequently the plane of projection is perpendicular to this diagonal of the cube.

In Fig. 338 are given the regular orthographic projections of the same cube. Let point a (a^h, a^v) be supposed to be the upper front corner; then b (b^h, b^v) will be the lower back corner, and an isometric projection may be obtained by projecting on any plane, such as Q (HQ, VQ), which is at right angles to the line ab. This isometric projection is of the same shape as the isometric drawing, but the projection of each edge of the cube is to its true length as $\sqrt{2}$ is to $\sqrt{3}$.

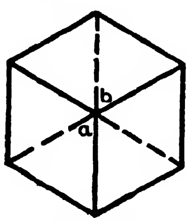

Fig. 337.

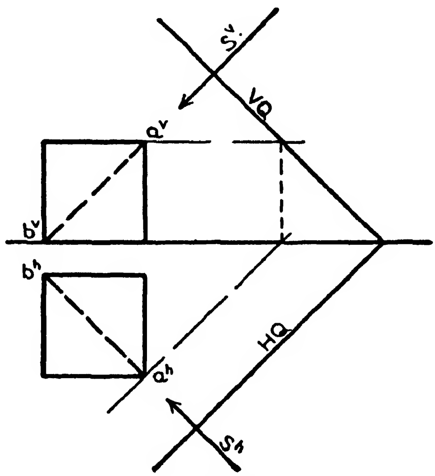

Fig. 338.

Note.—The proof of the above statement is based on the fact that the three edges of the cube which meet at point a make equal angles with the diagonal ab, and consequently with the plane Q which is perpendicular to ab. The rest of the proof is left to the student.

In making the isometric drawings of Fig. 337, we may therefore imagine that we are looking at the cube in the direction of the line S, Fig. 338, perpendicular to the plane Q.

The two projections of S are evidently inclined at an angle of 45° with the horizontal.

205. Relation between Orthographic Projection and Isometric Drawing. It is evident from Figs. 335 and 336 that the isometric drawing of a simple rectangular object may be made at once, without previous construction of any orthographic projections. In most cases, however, the isometric drawing wanted is that of some object already drawn in plan and elevation.

The isometric drawing of a simple object, given by its plan and elevation, is shown in Fig. 339. In accordance with Fig.

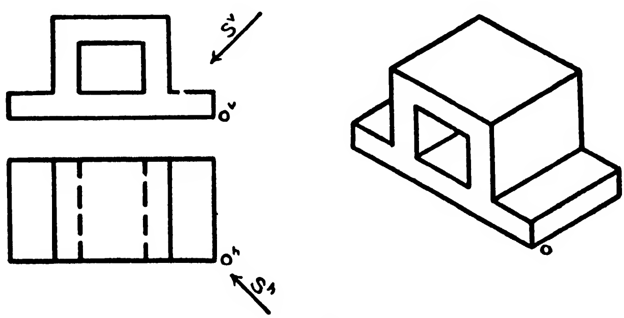

FIG. 339.

338, this object is considered as viewed in the direction of the arrow S (S^h, S^v), both of whose projections make an angle of 45° with the horizontal.

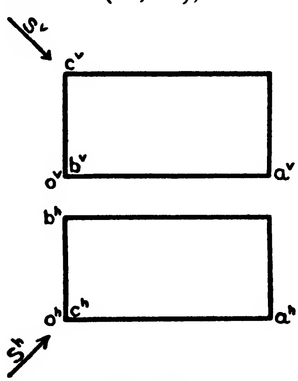

FIG. 340.

But while the projections of the rays of sight always make 45° with the horizontal, they need not always slope as in Figs. 338 and 339. This direction gives a view of the front, top, and right-hand side of the object. If the direction of sight be that of the arrow S (S^h, S^v), Fig. 340, we have a view of the front, top, and left-hand side of the object, as has already been given in Fig. 336.

206. Non-Isometric Lines. In the isometric drawings already given, every line of the object is parallel to one of the three isometric axes, and appears in its true length in the isometric drawing. But lines in other directions will not so appear. For instance, in the prism, Fig. 336, the two diagonals of the rectangular face AOC are equal in the object, but are not equal in the isometric drawing. Neither does the true angle between them appear in the isometric case.

Lines not parallel to any of the isometric axes are known as non-isometric lines. Since one of the principal uses of an isometric drawing is to represent objects which are more or less irregular, a number of non-isometric lines usually occur in every isometric drawing.

207. Isometric of an Irregular Object. In Fig. 341 an object is given by means of its front and side elevations (V- and P-projections, the P-projection being that of the right-hand side). This object may be considered as a block of iron with two of its upper edges chamfered or beveled off, and a triangular hole cut through.

To make the isometric drawing, we first draw the complete rectangular block indicated by the three axes OA, OB, OC. Points 1, 2, and 3 may be located from C, where the distance $C1 = c^p1^p$, $C2 = c^p2^p$, $C3 = c^p3^p$. Point 4 may be found by making $A4 = a^p4^p$. It will be seen that the distances measured are all in one of the isometric directions.

To locate the corner 5 of the triangular hole, we must first draw the rectangular coördinate 5^v9^v in the V-projection. Then in the isometric, make $O\text{–}9 = o^v9^v$, and $9\text{–}5 = 9^v5^v$. Point 6 in the isometric is similarly found by the coördinates O–10 and 10–6. Point 8 is in a non-isometric plane. Locate first point 7, in the isometric plane OBC. Then locate 8 from 7 by drawing 7–8 parallel to the isometric axis OA, and make the distance $7\text{–}8 = 7^p8^p$.

The isometric drawing may then be completed as shown.

208. Isometric Coördinates. Any point in space may be located from another point as origin by the use of not more

than three mutually perpendicular coördinates. If properly chosen, these rectangular coördinates may then be laid off in an isometric drawing in directions parallel to the isometric axes, and are then known as isometric coördinates. Hence any point in an isometric drawing may be located by the use of not more than three isometric coördinates. The use of

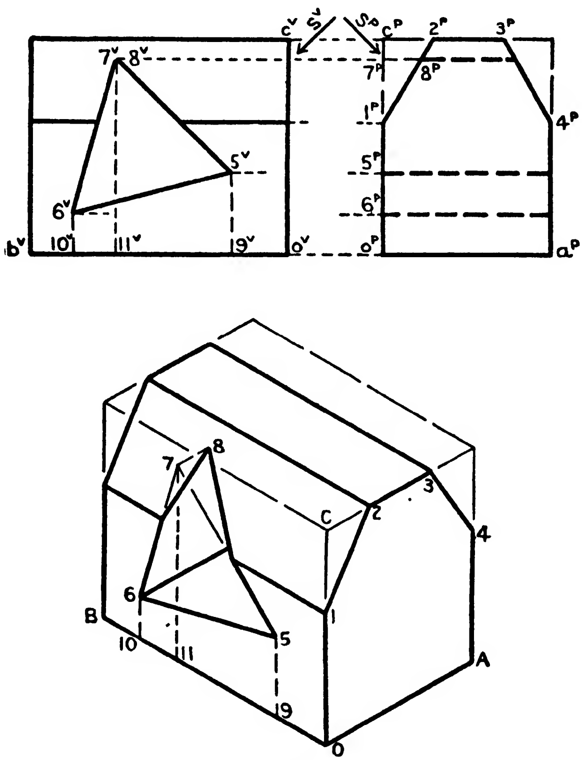

FIG. 341.

three coördinates has already been shown in the determination of point 8, in Fig. 341.

209. An Additional Example. As a further illustration of the use of isometric coördinates, let us make the isometric drawing of an irregular triangular pyramid, Fig. 342. Taking point o (o^h, o^v) in the projection views as the origin, draw

rectangular coördinates in these views. Then starting with point 0 in the isometric, point 1 is located by one coördinate 0–1; point 2 by two coördinates, 0–5 and 5–2; point 3 by three coördinates, 0–6, 6–7, 7–3; and point 4 by three coördinates, 0–8, 8–9, 9–4. Note how these coördinates are obtained from the plan and elevation.

The visibility of the isometric drawing is determined by considering the line of sight S in the orthographic projection. This shows point 3 to be the farthest from the observer. The rest is determined by the isometric drawing itself. The point 3

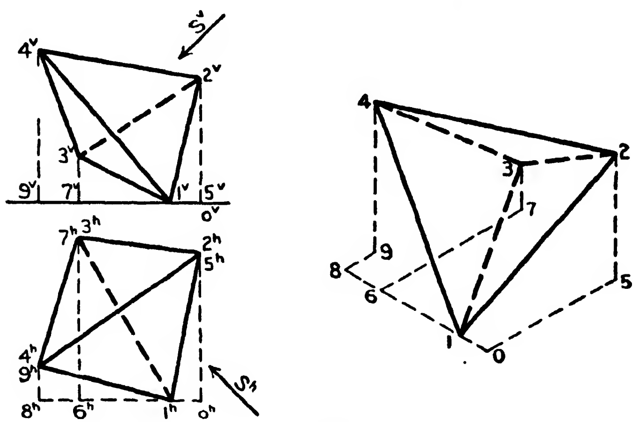

FIG. 342.

in the isometric falls inside the triangle 1–2–4, hence point 3 is invisible. That is, only the face 1–2–4 is visible. Had the coördinate 7–3 been greater, that is, had the point 3 been higher, so as to come above the line 2–4 in the isometric, the face 2–3–4 would also have been visible.

Since all the edges of this solid are non-isometric lines, the resulting isometric drawing is not especially satisfactory. It is given merely as an extreme case of the use of isometric coördinates.

210. Isometric Drawing of Curves. The isometric drawing of any curve may be obtained by locating a sufficient number of points in the curve by the use of isometric coördinates. The

curves which occur most frequently are circles lying in one of the isometric planes.

For circles of moderate size, it is sufficient to locate eight points equally spaced on the circumference. This is shown in Fig. 343. At the left is the front elevation of a circle lying in a vertical plane. At the right is the isometric drawing of the same circle. If the center of the circle be taken as the origin of coördinates, only two distances are needed to locate

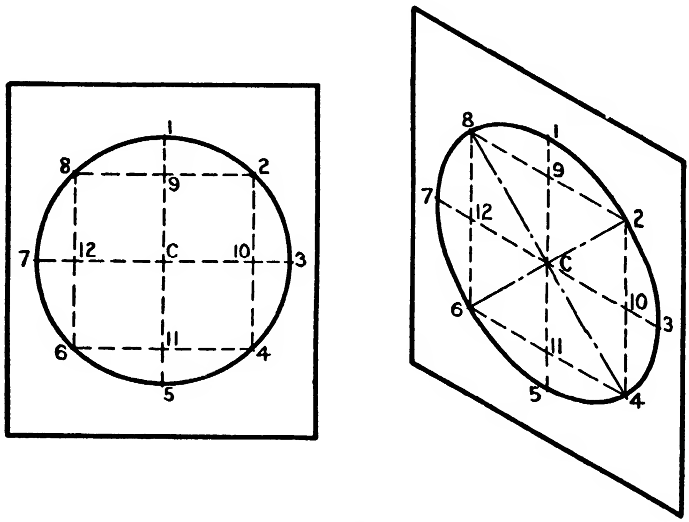

FIG. 343.

the eight points in the isometric, namely, $C1 = C3 = C5 = C7$ = the radius of the circle, and $C9 = C10 = C11 = C12$ = half the side of the inscribed square 2–4–6–8. The isometric drawing is an ellipse, in which 4–8 and 2–6 are the major and minor axes respectively.

A further illustration of the location of curves in an isometric drawing is given in Fig. 344. The object is the frustum of a circular cylinder. Point c, the center of the base, is taken as the origin of coördinates. By comparing the numbered points in the projection and isometric views, the coördinates used for each point should become readily apparent.

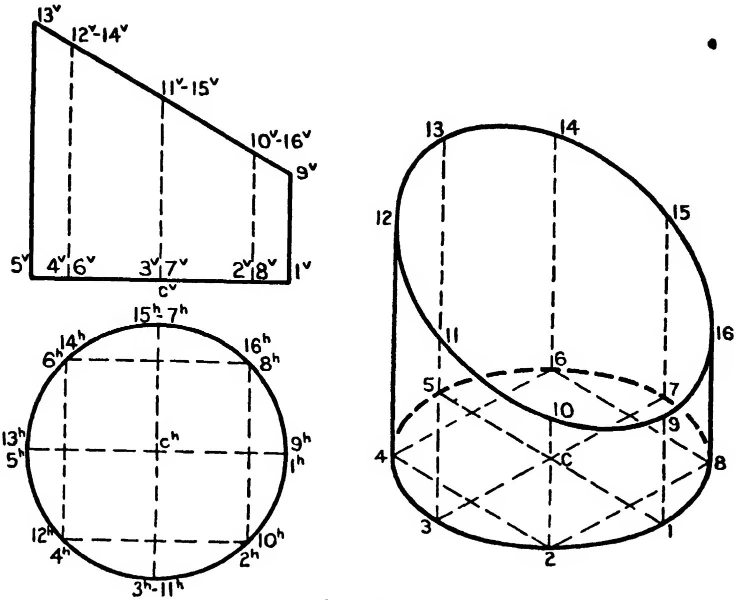

FIG. 344.

211. Approximations for Small Circular Arcs. In the case of quarter-circles used for rounded corners, approximations may be used if the radius of the circle (on the drawing) does not exceed one inch. These approximations are shown in Fig. 345.

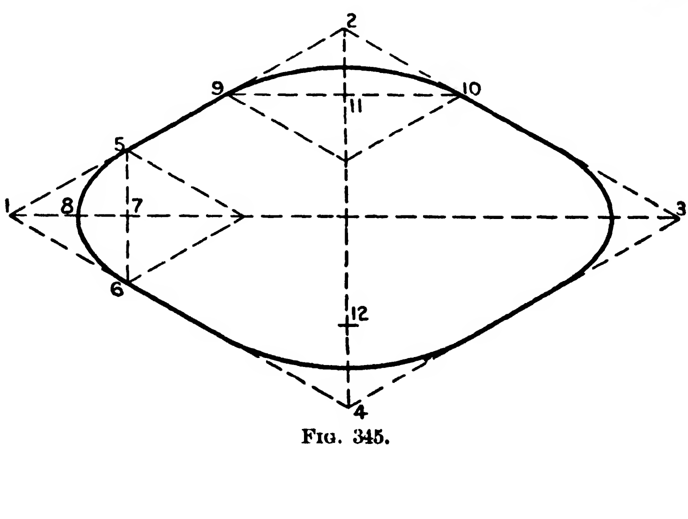

FIG. 345.

For a corner such as 2, lay off 2–9 = 2–10 = the radius of the circle. Draw 9–10. Make the distance 11–12 equal to the distance 9–10. Then with 12 as a center draw a circular arc 9–10. This arc is not strictly tangent to the straight lines at 9 and 10, but is nevertheless a very close approximation.

For a corner such as 1, lay off 1–5 = 1–6 = the radius of the circle. Draw 5–6. Make the distance 1–8 equal to 5–7 or 6–7. Then point 8 is located with an error less than one per cent of the distance 1–5. The curve 5–8–6 should be sketched. There is no single circular arc which is a good approximation to this curve, while a construction using three circular arcs becomes too small to be manageable.

212. Other Axonometric Systems. The construction of axonometric drawings other than isometric will not be considered in detail, since the method used, namely, the location of points by coördinates parallel to the axonometric axes, is essentially the same.

It may be remarked in passing, however, that systems of axonometric axes which make simple angles with each other, and at the same time bear simple ratios to the lengths of the lines which they represent, are not easy to find. The system obtained in § 202 is, next to the isometric system, probably the simplest. Another fairly simple system, with a good pictorial effect, is shown in Fig. 346, the object being the same square prism already given in Figs. 334 and 336. In Fig. 346 OC is vertical and equal to the true length of that edge of the prism; OA is inclined at 12° with the horizontal, and equal to the true length of OA; while OB is at 45° with the horizontal, and two-thirds the length of the line in the object. Even this system cannot be used unless a ready means for drawing lines inclined at 12° with the horizontal is available.

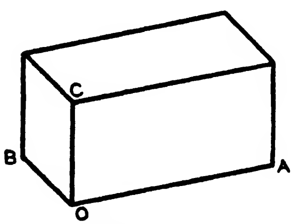

FIG. 346.

If for any reason an isometric drawing is not suitable, recourse is usually had to some form of oblique projection.

213. Oblique Projection. Let the same square prism previously made use of be placed with one of its faces in a vertical plane of projection. Looked at orthographically, that is, at right angles to the plane of projection, the result is the familiar one shown in Fig. 340. The four edges perpendicular to the plane of projection project merely as points. Instead, let the prism be looked at with rays of sight which are parallel, but at some direction oblique to the plane of projection. The front face of the prism, being in the plane of projection, will still appear in its true shape and size. But the edges perpendicular to the plane of projection will no longer project as points, but as parallel lines. Depending on the slope and obliquity of the rays of sight, these lines may have any direction, and be of any length, greater than, less than, or equal to, the true lengths of the edges themselves.

If we assume the direction of sight to be such that these edges project equal to their true length, and their projections inclined upward to the right at 45° with the horizontal, the resultant projection, omitting invisible lines, is shown in Fig. 347. The result is quite pictorial, and can be utilized as easily and in the same manner as axonometric drawing, by taking OA, OB, and OC as a system of coördinate axes.

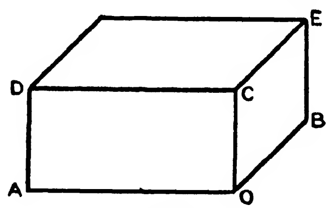

FIG. 347.

In this system of representation, the axes OA and OC are always at right angles to each other, but the axis OB may be inclined at any angle, usually for convenience at 45° or 30° with the horizontal. This axis may also be inclined either to the right or to the left, according to which side of the object it is desired to show. Further, in order to heighten the pictorial effect, the coördinates parallel to the axis OB are sometimes reduced to one-half or two-thirds of their actual length.

214. Comparison of Isometric Drawing and Oblique Projection. Under certain conditions, an oblique projection may be much more readily constructed than an isometric drawing, and at the same time present a better appearance. This is especially the case if there are a number of circles, circular arcs, or other curves lying in vertical planes parallel to the plane of the drawing. This is apparent in Fig. 348, where it will be noted that the circles and circular arcs appear in their true shape, and are readily constructed.

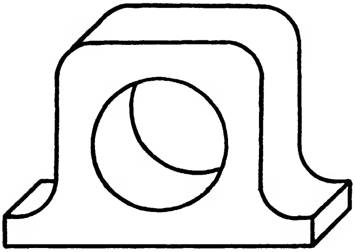

FIG. 348.

But circles lying in other positions, however, show as ellipses which must be constructed by points, so that there is no gain over an isometric drawing. In fact, the isometric drawing usually has the best of it in appearance. The ellipses constructed in oblique projection generally give a very distorted idea of the circles which they represent. Figs. 349, 350, and 351 are representations of the same hollow cylinder, its length being twice its outside diameter. Fig. 349 is the easiest to construct, and is fairly satisfactory. In Fig. 350 the circular end is in a vertical plane perpendicular to the plane of the paper, and appears very much distorted. The isometric drawing, Fig. 351, is the best representation of the three.

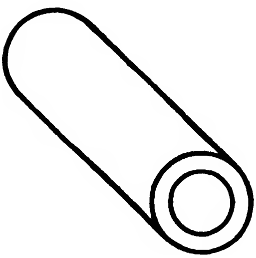

FIG. 349.

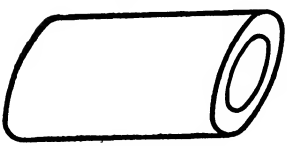

FIG. 350.

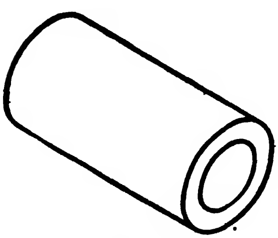

FIG. 351.

Another illustration is given by Figs. 352, 353, and 354. The object is a thin rectangular plate, punched through with

six circular holes. If the plate is placed in a vertical position, Fig. 352, the oblique projection is easy to construct, and has a good appearance. If the plate is placed in a horizontal posi-

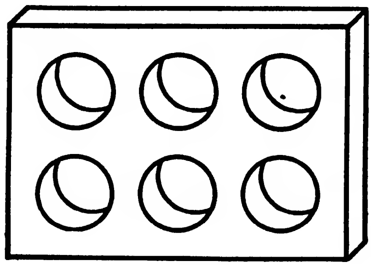

Fig. 352.

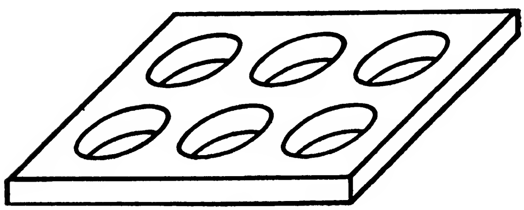

Fig. 353.

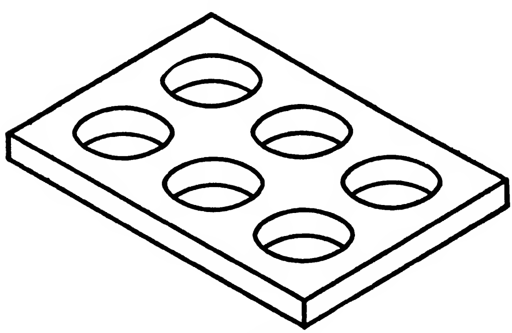

Fig. 354.

tion, the oblique projection, Fig. 353, is unsatisfactory. The holes do not appear to be circles lying in a horizontal plane. This objection does not hold, however, against the isometric drawing, Fig. 354.

215. Distortion. In a true pictorial (perspective) representation, parallel lines which recede from the observer are represented by lines which converge to a point. In all of the foregoing pseudo-pictorial systems, such lines are represented by parallel lines in the drawing. This introduces a certain apparent distortion which cannot be avoided, but which can be minimized by omitting all the invisible lines of the drawing. (See § 202.)

Other apparent distortions can often be overcome by changing the position of the object relative to the observer, that is, by changing the point of view, as has been shown in § 214. With some objects, especially objects containing curved lines or lines not parallel to one of the three rectangular axes, it may be necessary to try, from several points of view, both isometric and oblique projections of the object, before a satisfactory representation is obtained.

CHAPTER XXVIII

WARPED SURFACES

216. Warped Surfaces. A warped surface is formed by a moving straight line, so guided that no two consecutive positions * lie in the same plane. That is, it is a ruled surface (§ 172) in which no two consecutive rulings are either intersecting or parallel. It is therefore a non-developable surface (§ 179).

217. Generation of Warped Surfaces. The moving straight line, or **generatrix**, is usually guided by lines, known as the **linear directrices.** These directrices may be straight lines, plane curves, or space curves (§ 150).

In choosing directrices, certain conditions must be observed. For instance, straight lines or curves lying in the same plane cannot be taken, since they would only guide the moving line to form this plane. Further, it is evident that two linear directrices are not sufficient to compel the generating line to assume a fixed position in space, since a straight line can be drawn from any point of either directrix to any point of the other. A third condition must be imposed. This condition may be a third linear directrix; or it may be a plane, called the plane directrix, to which all positions of the generatrix must be parallel. Other methods of generation will be given in connection with the surfaces discussed.

218. Singly and Doubly Ruled Surfaces. Let a warped surface be generated with three *straight* lines as linear directrices. Take any three positions of the generatrix as a second set of linear directrices. Then the same surface may be formed by the motion of a second generatrix, of which the original directrices are three positions. Hence, through any point of the surface two distinct rectilinear elements may be drawn, one

* Strictly speaking, consecutive lines do not exist on the surface; but it is convenient to regard a surface as the limiting form approached by lines that are consecutive.

corresponding to each method of generation. Such a surface, containing two distinct sets or systems of elements, is doubly ruled. An important feature of the doubly ruled surface is that every element of either system intersects every element of the other system.

Most warped surfaces, however, cannot be formed by the exclusive use of *straight* lines as directrices, and are singly ruled. In general, through any point in a singly ruled surface but one rectilinear element can be drawn.

219. Representation of Warped Surfaces. Since a straight line is indefinite in extent, a surface formed by it is also indefinite in extent. A warped surface is usually represented by showing the directrices and a number of elements contained in a limited portion of the surface.

220. Tangent Planes. The general principles of §§ 154 and 155 apply for the surfaces of this chapter. Through any point in a warped surface, a rectilinear element can be drawn, which must lie in the tangent plane at that point (§ 155). If the surface is doubly ruled, the tangent plane is the plane determined by the two elements which pass through the point.

(*a*) A plane tangent to a warped surface is, in general, tangent at only a single point, and elsewhere secant to the surface. Conversely, any plane that contains an element of a warped surface will, in general, be tangent to the surface at some point.

(*b*) If a straight line intersects a warped surface, a plane tangent to the surface may, in general, be passed through the given line. For, through the point in which the line intersects the surface, draw the rectilinear element. The plane of these two lines, since it contains an element, will, in general, be tangent to the surface at some point.

If the surface is doubly ruled, two elements can be drawn through the point in which the given line intersects the surface. Hence two tangent planes can generally be found.

221. The Intersection of a Warped Surface and a Plane. The general method has been given in § 173. Points are obtained

by finding where elements of the surface intersect the given plane.

A line tangent to the intersection at any point is found as explained in § 178. The tangent line is the intersection of the given secant plane with the plane tangent to the surface at the given point. This method fails only at a point where the given plane is itself tangent to the surface, if such a point exists.

THE HYPERBOLIC PARABOLOID

222. Generation. The **hyperbolic paraboloid** has two rectilinear directrices and a plane directrix. In Fig. 355, A and B are the rectilinear directrices, and Q the plane directrix, to

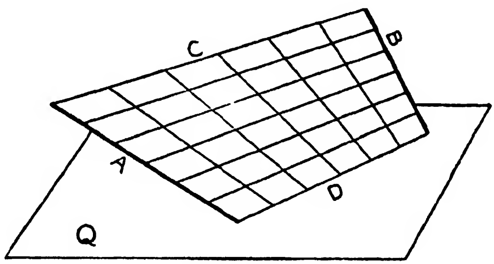

FIG 355.

which the elements, as C and D, are parallel. The same surface is shown in projection in Fig. 356, where H is substituted for Q as the plane directrix.

Elements of the surface may also be drawn by dividing the directrices A and B into the same number of equal (or proportional) parts, and connecting the points of division.

A second set of elements can be obtained by taking C and D as directrices. These elements are parallel to a plane directrix (not shown) which is parallel to A and B. The surface is thus doubly ruled.

223. Forms of Plane Sections. Any plane will, in general, intersect the hyperbolic paraboloid in either a parabola or a hyperbola, whence its name. But a plane parallel to a plane

directrix will cut the surface in a straight line. Thus, the plane *X*, Fig. 356, is parallel to both *A* and *B*; hence it is

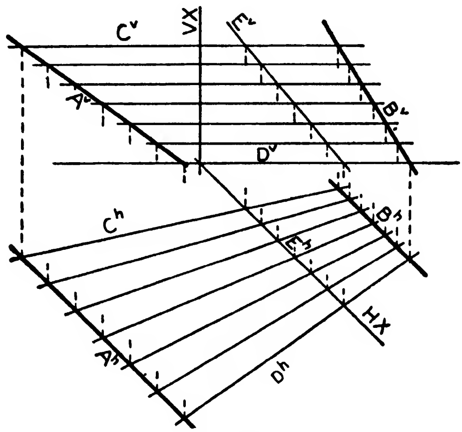

FIG. 356.

parallel to the plane directrix which is parallel to *A* and *B*. Plane *X* intersects the surface in the straight line *E*, an element of the surface.

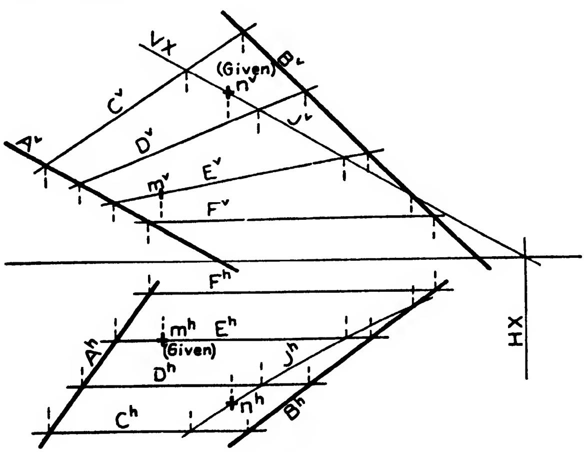

FIG. 357.

224. To Project a Point in the Surface. In Fig. 357, *A* and *B* are taken as linear directrices, and *V* the plane directrix.

Let m^h be given. Since one set of elements is parallel to V, through m^h draw the H-projection, E^h, of an element parallel to V. Find E^v from the intersection of E with the directrices A and B. Then m^v lies in E^v.

Let n^v be given. Draw several elements of the surface, as C, D, and E. Through n^v pass any plane X perpendicular to V. Find the intersection, J, of this plane with the surface, by noting the points where X cuts the elements C, D, E. Then n^h lies in J^h.

225. Symmetry of the Surface. Although it is not apparent in Figs. 356 and 357, the hyperbolic paraboloid has an axis of symmetry. This is shown by placing the surface as in Fig. 358, where A and B, C and D, are the two sets of linear

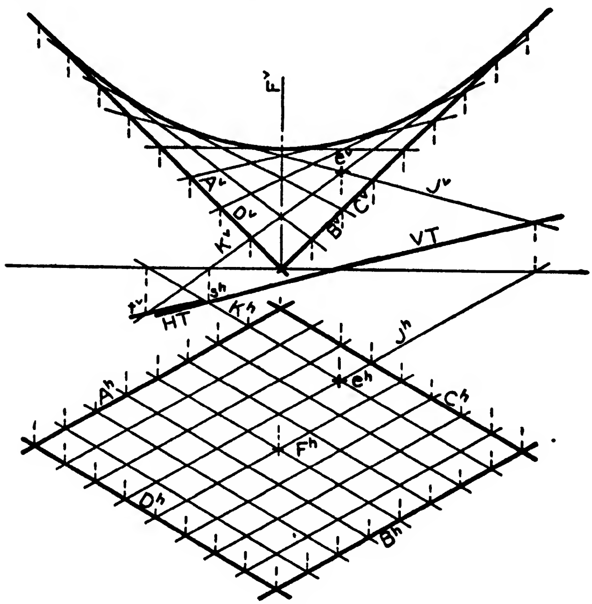

FIG. 358.

directrices. The elements are here obtained by dividing each directrix into the same number of equal parts. The contour of the surface in the V-projection is the envelope of the projected elements, a parabola, and F is the axis of symmetry.

Problem 53. ***To pass a plane tangent to a hyperbolic paraboloid at a given point in the surface.***

Analysis. The plane is determined by the two elements (one of each system) which pass through the given point (§ 220).

Construction (Fig. 358). Let e be the given point. The required tangent plane T contains the two elements, J and K, which pass through e.

Problem 54. ***To pass a plane tangent to a hyperbolic paraboloid through a given line which intersects the surface.***

Analysis. See § 220 *b*.

Construction (Fig. 359). Let A and B be the linear directrices and H the plane directrix of the hyperbolic paraboloid.

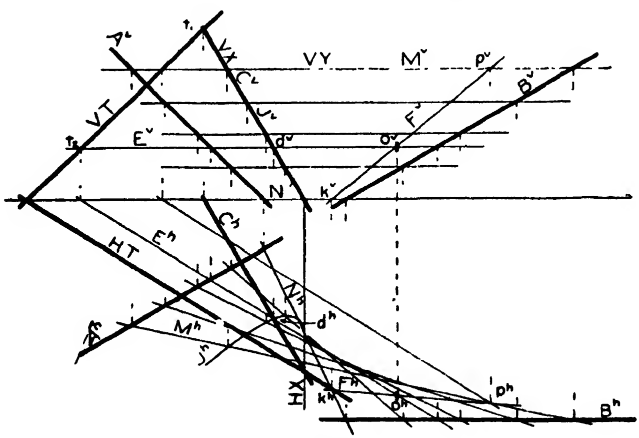

FIG. 359.

Let C be the given line. Draw a number of elements of the surface, as shown. Through C pass the plane X perpendicular to V. Find the intersection, J, of X and the surface. Where J crosses the line C gives the point, d, in which C intersects the surface. Through d draw the element E. Pass the required tangent plane, T, through the given line C and the element E.

To find the point of tangency of plane T. Since the surface is doubly ruled, the tangent plane T will contain two

elements, one of each system of generation (§ 220). Hence, in addition to the element E, a second element F lying in T may be found. The element F is determined by connecting the points p and k, in which any two elements of the first system, as M and N, intersect the plane T. The intersection of E and F gives O, the required point of tangency.

NOTE. Since two elements can be drawn through the point d, two tangent planes through the given line C are possible (§ 220 b). The second element is omitted in Fig. 359. It may be found, if desired, by passing through d a plane parallel to the lines A and B, and then finding the intersection of this plane with the surface.

Problem 55. ***To find the intersection of a hyperbolic paraboloid and a plane.*** ***(General Case.)***

Analysis. Points are obtained by finding where elements of the surface intersect the given plane (§§ 173, 221).

Construction (Fig. 360). Let A and B be the linear directrices, and V the plane directrix of the hyperbolic paraboloid.

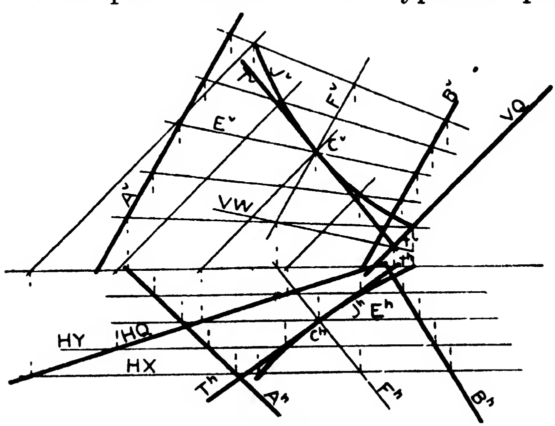

FIG. 360.

Let Q be the cutting plane. Assume elements of the surface as shown, and find the points in which they intersect the plane Q. This gives the required curve J. In finding these points, take the auxiliary planes, as X, Y, etc., perpendicular to that coördinate plane on which the elements project as parallel lines, thus obtaining parallel lines of intersection.

To draw a tangent to the intersection J at the point c on element E (§ 221). Through c draw a second element F, its V-projection being parallel to A and B. The plane containing E and F is tangent at the point c. Its V-trace, VW, is readily found. The intersection of VW and VQ is the V-trace of the required tangent line T.

THE UNPARTED HYPERBOLOID OF REVOLUTION

226. Generation. The **unparted hyperboloid of revolution,** or hyperboloid of revolution of one nappe, may be formed by the revolution of a hyperbola about its conjugate axis. It may

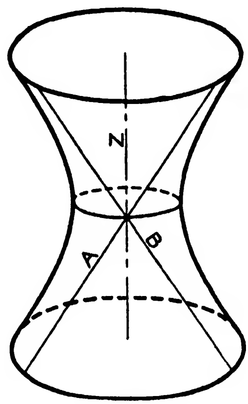

FIG. 361.

also be formed by the revolution of one straight line about another not in the same plane. The surface is doubly ruled: thus, in Fig. 361, lines A and B make equal angles with the axis Z, and either will generate the surface.

227. Circle of the Gorge. Any point in the generating line describes a circle lying in a plane perpendicular to the axis, a parallel of the surface (§ 166). The point in the generating line which is nearest the axis describes the smallest circle, called the circle of the gorge.

228. Representation of the Surface. It is customary to represent only a limited portion of the surface bounded by two parallels which are equidistant from the circle of the gorge.

(*a*) Representation by Means of Elements (Fig. 362). Let the axis of the surface be taken perpendicular to H. In the H-projection the elements will all appear tangent to the circle of the gorge, which shows in true size. In Fig. 362, the elements are taken equally spaced around the circle of the gorge, and the symmetry of the surface and its double ruling

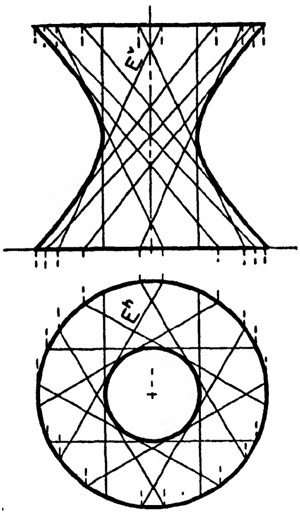

Fig. 362.

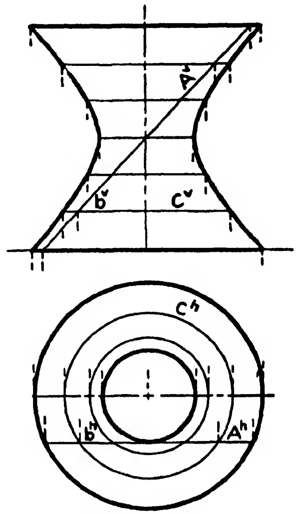

Fig. 363.

are readily apparent. The hyperbolic outline in the V-projection is obtained as the envelope of the elements. Any element, as E, may be considered to be the generatrix.

(*b*) Representation by Means of Parallels (Fig. 363). Let the axis of the surface be perpendicular to H, and A the generating line. Any point in A, as b, will describe the circle C, a parallel of the surface. The hyperbolic outline of the V-projection is thus obtained as a series of points. If the generating line A be taken parallel to V, as shown, its V-projection will be one of the asymptotes of the hyperbola.

229. To Project a Point in the Surface. Let the axis of the hyperboloid, Fig. 364, be taken perpendicular to H.

(*a*) Let a^h be the given projection. Through a^h draw the H-projection, E^h, of an element, tangent to the circle of the

gorge. Find the V-projection, E^v, of this element, which must contain the required projection, a^v.

Since the element drawn in the H-projection may also represent a symmetrically placed element F (F^h, F^v), there is a second point b (b^h, b^v) corresponding to the given H-projection.

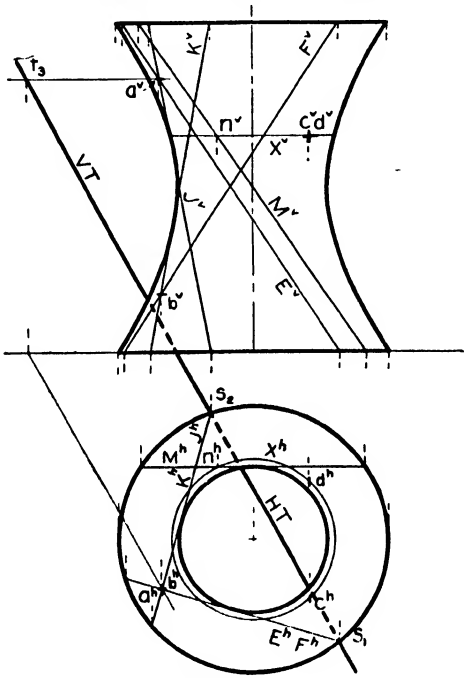

FIG. 364.

The two points a and b can also be projected by the use of the two elements J and K, whose H-projections are also tangent to the circle of the gorge. If both sets of elements be used, the projections a^v and b^v will each be found at the intersection of the V-projection of two elements.

(b) Let c^v be given. Through c^v draw X^v, a parallel of the surface. The H-projection of the parallel is the circle X^h. To obtain X^h, note the point, n^v, where X^v intersects the V-projection, M^v, of the generating line (or any element). Project n^v to n^h in M^h; then X^h passes through n^h. Project

from c^v to the required point, c^h in X^h. There is evidently a second point, d (d^h, d^v), corresponding to the given V-projection.

Problem 56. ***To pass a plane tangent to an unparted hyperboloid at a given point in the surface.***

Analysis. Since the surface is doubly ruled, the plane is determined by the two elements which pass through the given point (§ 220).

Construction (Fig. 364). Let a (a^h, a^v) be the given point. Draw the two elements, E and J, which pass through this point (§ 229 a). Pass the required tangent plane, T, through the lines E and J.

Problem 57. ***To pass a plane tangent to an unparted hyperboloid of revolution through a given line which intersects the surface.***

Analysis. See § 220 b.

Construction (Fig. 365). Let A be the given line. Let the hyperboloid be formed by the revolution of the line B about an axis perpendicular to H. The surface is here shown constructed by the method of parallels (§ 228 b). It might equally well be formed by the method of elements (§ 228 a), but the elements used in the subsequent construction would then not be so readily apparent.

Find the point c in which the given line A intersects the surface. To do this, pass through A an auxiliary plane; here the plane X, perpendicular to H. This auxiliary plane cuts from the surface a curve, whose V-projection, R^v, intersects A^v at c^v, one projection of the point in which A intersects the surface.

Through c draw the element E of the surface (§ 229 a). Through the lines A and E pass the plane Q, one of the required tangent planes.

Draw also the other element F which passes through c. Through A and F pass the second required tangent plane T.

To find the points of tangency of the two planes. The plane Q contains a second element J, the H-trace of which is at s_4, where HQ cuts the parallel of the surface which lies in H. Through s_4 draw J^h tangent to the circle of the gorge. Since two tangents can be drawn from s_4, we must select for

J^h the one symmetrical with E^h, that is, J^h and E^h must make equal angles with HQ. From J^h find J^v. Then the elements E and J both lie in the plane Q. They intersect at point o, which must be the point of tangency of the plane Q (§ 220).

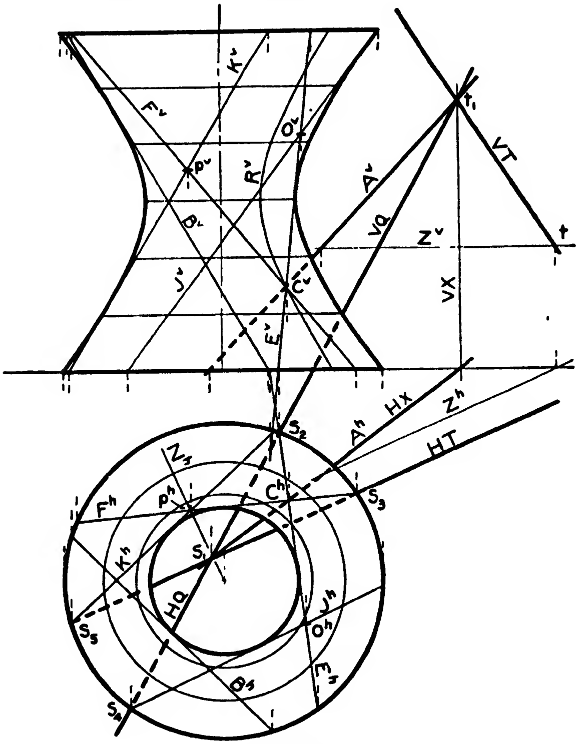

FIG. 365.

Similarly, the plane T contains, besides the element F, the element K, obtained primarily from the intersection s_5 on HT. Elements F and K intersect at point p, the point of tangency of plane T.

An additional line passing through point p can be readily obtained in the H-projection. For the normal to any point in the surface must intersect the axis (§ 170 e), and the normal at the point p is perpendicular to the tangent plane at that point. If, then, through the axis of the surface we draw N^h

perpendicular to HT, N^h will pass through p^h. This may be used to insure accuracy, if the intersection of F^h and K^h is flat; or the intersection of F^h and N^h used to locate p^h, without drawing K^h.

NOTE. The line A and the hyperboloid both being indefinite in extent, there will be, in general, a second intersection of A with the surface. Through this point two elements may be drawn, each of which will determine a tangent plane. But there are not four planes which can be passed through the given line tangent to the surface. The elements which pass through the second point in which line A intersects the surface are, in fact, the elements J and K already drawn, and by their use we shall merely obtain the same two planes Q and T. The two possible tangent planes may be found by the use of either point in which the given line intersects the surface.

Problem 58. ***To find the intersection of an unparted hyperboloid of revolution and a plane.***

First Analysis. Points are obtained by finding where elements of the surface intersect the given plane (§§ 173, 221).

Second Analysis. The surface may be treated as a surface of revolution, and the intersection found accordingly. See § 182 and the First Construction of Problem 45.

Let the surface be given as shown in Fig. 366, and Q the cutting plane.

Construction by First Analysis. This is illustrated by the elements E and F. The points, 1 and 2, in which these lines intersect the plane Q are found by using the auxiliary plane Z, perpendicular to H. Other points may be found in a similar manner. With this surface, this construction is not so convenient as the following method.

Construction by Second Analysis. The plane X, perpendicular to the axis of the hyperboloid, intersects the surface in the circle C, and the plane Q in the line B. These intersect in points 3 and 4, two points in the required intersection. Other points may be found in a similar manner. (Compare Fig. 311.)

A. POINTS DETERMINED BY MERIDIAN PLANES. In addition to the points obtained by either of the above methods, there should always be obtained the points that lie in the meridian

plane of symmetry, M, and the principal meridian plane Y, as explained in detail in connection with Problem 45. The construction here is as follows:

The principal meridian plane Y intersects the plane Q in the line J. The V-projection, J^v, intersects the hyperbolic outline in points 5^v and 6^v, the V-projections of points in the curve of intersection. The V-projection of the curve of intersection is tangent to the outline of the surface at 5^v and 6^v, and changes visibility at these points. The intersection of

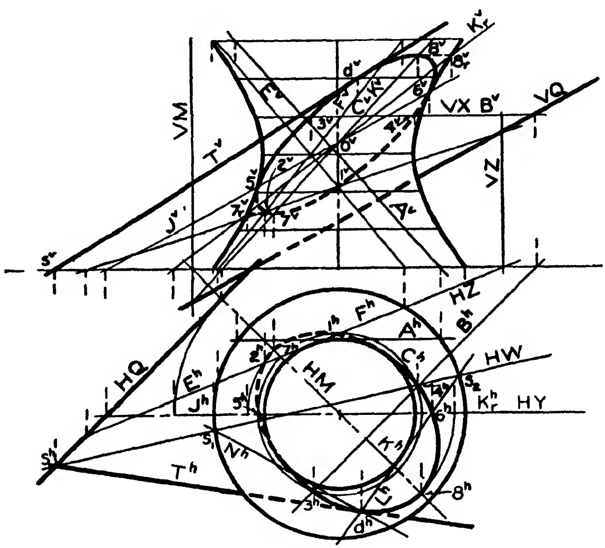

FIG. 366.

J^v and the axis gives the V-projection, o^v, of the point in which the plane Q intersects the axis.

The meridian plane of symmetry M intersects the plane Q in the line K. This line is revolved about the point o into the principal meridian plane of the surface. In the V-projection the revolved position K_r^v intersects the contour of the surface in the points 7_r^v and 8_r^v. These are the revolved positions of two points in the intersection, whose projections, 7^v, 8^v, and 7^h, 8^h are found by counter-revolving the line K. In this case these points are the lowest and highest points in the curve of

intersection. (Compare the same construction, lines J and K, in Fig. 311.)

B. A LINE TANGENT TO THE CURVE OF INTERSECTION (§§ 178, 221). Let d be the given point. Pass the plane W, tangent at the point d (Prob. 56). Only the H-trace, HW, is needed, so only the H-projections, L^h and N^h, of the elements passing through d need be drawn. The intersection of HW and HQ gives the H-trace, s, of the required tangent line T, which may now be drawn connecting s and d.

NOTE. The intersection of an unparted hyperboloid of revolution and a plane is a conic, and may be either a circle, an ellipse, a parabola, an hyperbola, two intersecting straight lines, or two parallel straight lines.

230. An Application Problem. We are now ready to consider the general solution of the following problem, in which the unparted hyperboloid of revolution is employed as an auxiliary to the solution. Special cases, not requiring the use of the hyperboloid, have been given in Problem 40.

Problem 59. ***To pass a plane tangent to a double curved surface of revolution through a given line.*** ***(General Case.)***

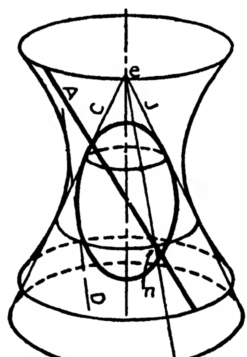

FIG. 367.

Analysis. (See Fig. 367.) Let the ellipsoid be the given double curved surface of revolution, and A the given line. Revolve the line A about the axis of the ellipsoid, thus generating a coaxial hyperboloid of revolution. In any common meridian plane of the ellipsoid and hyperboloid, draw a line C tangent to the meridian curve of each surface. The line C, lying in a plane containing the common axis of the surfaces, will intersect the axis at some point, as e. (In rare instances, C may be parallel to the axis.) Revolve the line C about the common axis. The line will then generate a cone (cylinder) of revolution. Pass the required tangent plane through the line A tangent to this cone (cylinder).

In general, more than one plane can be passed through the given line tangent to the double-curved surface of revolution. Thus, in Fig. 367, the line D is also tangent to the meridian sections of the ellipsoid and hyperboloid. The revolution of D will generate a second cone, and a plane through A tangent to this cone will also be tangent to the given ellipsoid. On the other hand, a common tangent line J, symmetric with C,

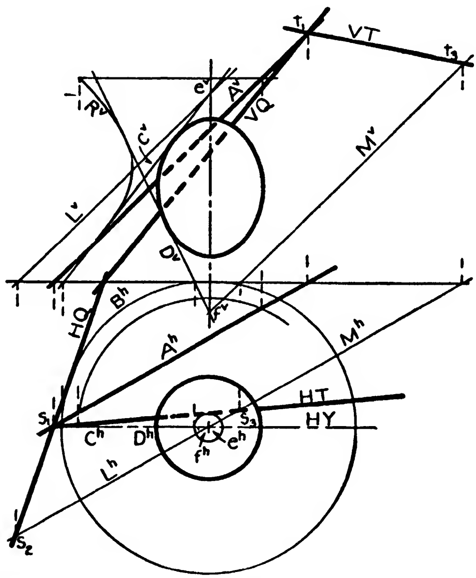

FIG. 368.

would not give a cone different from one already formed, and therefore would not give an additional answer to the problem.

Proof of the Analysis. Since the cone formed by the revolution of the common tangent line C circumscribes the given double-curved surface of revolution, it is evident that any plane tangent to this cone is also tangent to the given double-curved surface. It is not so evident, however, that a plane can be passed tangent to the cone through the given line A, for, in general, a plane cannot be passed tangent to a cone

through a line. The line A is one element of a surface which circumscribes and is tangent to the cone. Hence the line A is tangent to the cone. Suppose a plane passed tangent to the cone at the point n where A is tangent. This plane must contain all lines tangent to the cone at the point n (§ 154); hence it must contain the line A. Thus it is possible for a plane tangent to the cone to contain the line A.

Construction (Fig. 368). The given surface is an ellipsoid with its axis perpendicular to H, and A is the given line. Revolve the line A about the axis of the ellipsoid, and construct the meridian section of the resulting hyperboloid of revolution at R^v, as shown (§ 228 *b*). Only one branch of the hyperbola need be drawn. Draw C^v tangent to both the hyperbola and the ellipse. The line C lies in the principal meridian plane, Y, of the ellipsoid, and the H-projection, C^h, coincides with HY.

It is now necessary to pass through the line A a plane tangent to the cone formed by the revolution of C about the axis of the given surface. Let C, as shown in the figure, intersect the axis at the point e (e^v, e^h), which must be the vertex of the cone. Then, since every plane tangent to the cone must contain the vertex (§ 162 *d*), pass the required plane Q through line A and the point e.

Otherwise, if the point e, the vertex of the cone, is not available, we can find the base of the cone on H. This is shown in the figure as the circle B^h. Then HQ passes through s_1, the H-trace of A, and is tangent to B^h (§ 162 *c*). This method must be used with caution. Two tangents can be drawn from s_1 to B^h. Only one of these can be the H-trace of a *tangent* plane which contains the line A, as there is but one such plane. (See Fig. 367 and the Analysis.) We must satisfy ourselves, by visualization or otherwise, that a tangent plane, whose trace is HQ, actually contains the line A, and is not on the opposite side of the cone from the line.

For a second result, draw the common tangent D (D^v, D^h). This locates the vertex of a cone at f (f^v, f^h). Pass the required plane T through the line A and the point f.

THE HELIX

231. The Helix. An important class of warped surfaces, known by the general name of helicoids, depends upon the use, as a directrix, of a space curve called the **helix**. Before considering any of the helicoids, it is therefore necessary to study the properties of the helix.

For present purposes, the most useful definition of the helix is that it is a curve drawn on the surface of a cylinder of revolution, making a constant angle with the elements. Hence, when the cylinder, or any portion of it, is developed, the helix will develop into a straight line. The distance measured along any element, between two successive points in which the helix crosses this element, is called the **pitch** of the helix.

232. Projection of the Helix. Let a cylinder of revolution, Fig. 369, on which a helix is to be drawn, be placed with its axis vertical, base on *H*. Let the length of the cylinder be equal to the pitch of the helix; then the cylinder will contain just one turn of the helix.

Since the curve is on the surface of the cylinder, the *H*-projection of the helix will be the circle which represents the cylinder. Starting with any given point, 0, divide the circle into any number of equal parts. In the *V*-projection, divide the length of the cylinder (equal to the pitch of the helix) into the same number of equal parts. Then project corresponding points of division, as shown. The *V*-projection of the helix is a sinuous curve.

233. Right-handed and Left-handed Helices. If the cylinder in Fig. 369, together with the helix drawn on its surface, be turned end for end, so that point 12 coincides with the present position of point 0, then 11 will be found at 1, 10 at 2, and so on. Indeed, any part of the helix has exactly the same curvature as any other part, and equal lengths of the curve can be made to coincide. The curvature of this helix is called right-handed. This is an arbitrary convention. With the axis of the helix vertical, as in Fig. 369, the fact that it is a right-

handed helix may be recognized by noting that the front of the curve (points 2, 3, 4) slopes upward to the right.

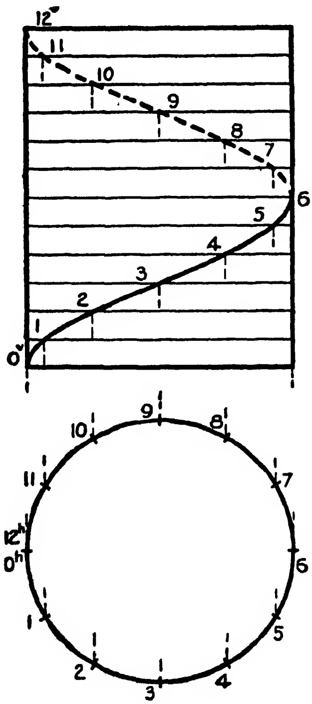

FIG. 369.

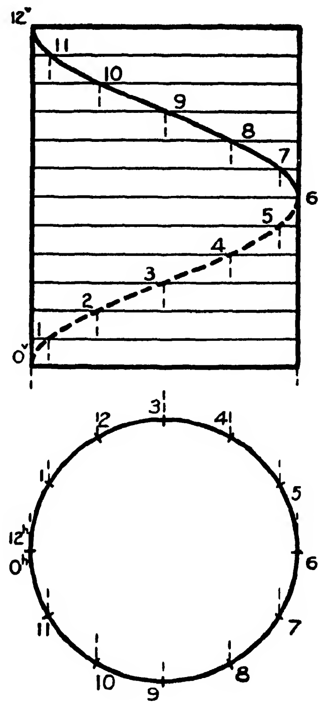

FIG. 370.

Since the curvature is constant, a right-handed helix remains right-handed, however placed. To get a left-handed helix, we must draw our curve around the cylinder in the opposite direction, as shown in Fig. 370. This helix may be recognized as left-handed from the fact that the front of the curve (points 8, 9, 10) slopes upward to the left.

Problem 60 ***To draw a line tangent to a helix at a given point in the curve.***

Analysis. The tangent may be obtained by developing a portion of the cylinder on which the helix lies, since the de-

veloped curve is a straight line (§ 231). Let a, Fig. 371, be the given point. Draw ab, the element of the cylinder passing through a. Pass a plane tangent to the cylinder along ab, and let the cylindrical surface to the left of ab be developed on this plane. Let the helix intersect the base of the cylinder at o. Then the arc bo of the base will develop into the

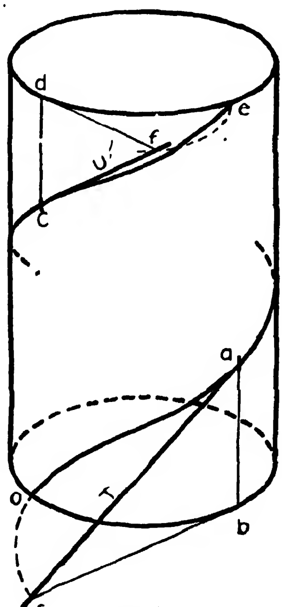

FIG. 371.

straight line bs, tangent to the base at b, and the length bs will be equal to the length of the arc bo. The curve oa will therefore develop into the straight line sa, which will be tangent to the helix at a.

In Fig. 371, the point c (on a different helix) is supposed to be so far above the base of the cylinder that it is not convenient to develop the length of the curve from c to the point on the base. We may therefore develop the portion of the helix between c and any convenient point e, as shown.

Construction (Fig. 372). Given three-quarters of a turn of a right-handed helix with a vertical axis. The cylinder on

which the curve lies being omitted, the entire curve is visible. Let a be the given point. At a^h draw T^h tangent to the circle which is the projection of the helix. Let us develop the helix from a to point o, which lies in H. Let ab be a vertical line. On T^h lay off the distance $b^h s^h$ equal to the length of the arc $b^h o^h$. Project from s^h to s^v in GL. Draw T^v through s^v and a^v.

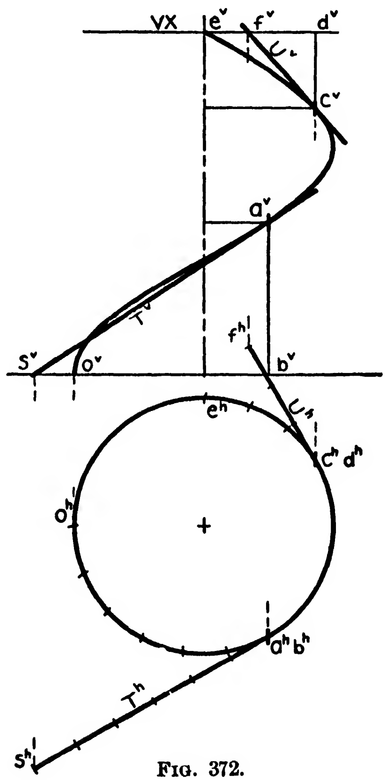

FIG. 372.

Let c be a second given point. Let it be decided to develop the helix from c to point e, lying in the plane X. At c^h draw U^h, tangent to the projection of the helix. On U^h make the distance $c^h f^h$ equal in length to the arc $c^h e^h$. Project f^h to f^v in VX. Draw U^v through f^v and c^v. Line U is tangent at the point c.

THE RIGHT HELICOID

234. Generation of the Surface. A **right helicoid** is formed by a moving straight line which is guided by a helix as directrix, and which remains at a constant distance from, and at right angles to, the axis of the helix. In the simplest form of the surface, the only one which will be considered here, the distance from the moving line to the axis of the helix is zero, that is, the generatrix constantly intersects the axis.

The surface derives its name of *right* helicoid from the right angle between the generatrix and axis, and not because of the kind of helix employed as directrix. A right helicoid can be formed equally well from a right-handed or a left-handed helix.

235. Representation of the Surface. As both the directing helix and the generating line are indefinite in extent, the surface is likewise, and only a very limited portion can be shown. The surface is evidently composed of two equal nappes, separated by the axis of the helix, and we may project either one nappe or both. In Fig. 373, the directing helix is supposed to be the numbered one, and the second nappe is shown by locating a similar helix on the opposite side of the axis. It is evident that any point in the generating line B, for example the point c, will describe a helix lying in the surface. Consequently, through any point in the surface a helix lying in the surface can be drawn.

Problem 61. *To pass a plane tangent to a right helicoid at a given point in the surface.*

Analysis. As this is a singly ruled surface, the plane will be determined by the element which contains the given point, and by the line tangent to the helix of the surface which passes through the point (§ 155).

Construction (Fig. 373). Let the point a, lying in the element E, be the given point. Draw the H-projection, the circular arc a^hc^h, of the helix in the surface passing through the point a. The V-projection of this helix, although shown in the figure, is not necessary. Draw the line J, tangent to this

helix at point a (Prob. 60). Pass the required plane, T, through the lines E and J.

Problem 62. ***To find the intersection of a right helicoid and a plane.***

Analysis. Points are obtained by finding where elements of the surface intersect the given plane (§§ 173, 221).

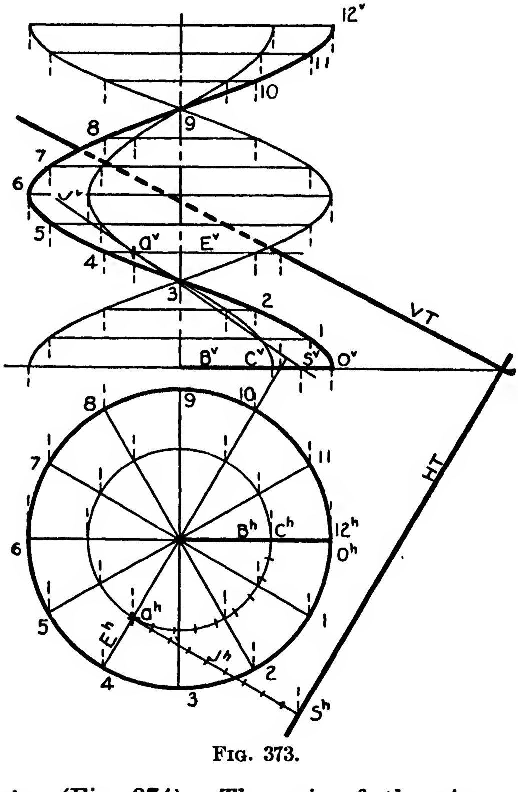

FIG. 373.

Construction (Fig. 374). The axis of the given surface is perpendicular to H. The elements of the surface are therefore parallel to H. In finding the intersection of this surface by the plane Q, it is preferable to pass the auxiliary planes through the V-projections of the elements. The lines of intersection of these planes and Q will then all be parallel. A typical plane is X (VX), locating the point a on the element 4.

A. To Find the Asymptotes. Let us pass the auxiliary plane *Y* through the element 5. The line of intersection, *M*, of this plane and *Q* is found to be parallel to the element 5. The line *M* then lies in the plane *Q*, and yet can contain no

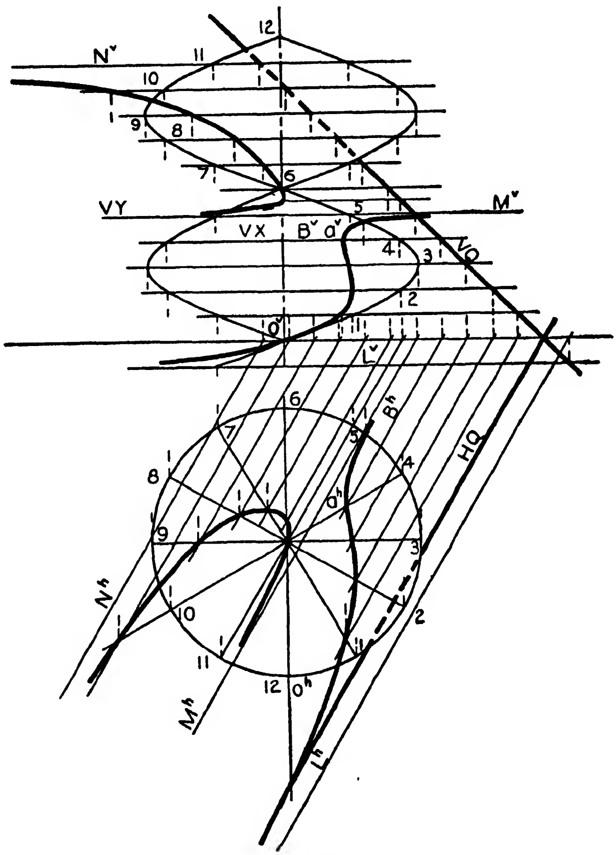

Fig. 374.

point lying in the given surface. The intersection of plane *Q* and the helicoid therefore has infinite branches, and *M* is an asymptote.

In fact, it is evident that in every complete turn of the surface there are two elements parallel to *Q*, and that in a surface

of indefinite extent these parallel elements will occur at regular intervals. A complete intersection of plane Q with the helicoid will consist of an indefinite number of infinite branches, separated by asymptotes at regular intervals. Two other asymptotes, L and N, are shown in Fig. 374.

This is the general form of the intersection of a right helicoid and a plane.

236. Special Sections of the Right Helicoid. Any plane containing the axis of a right helicoid will cut the surface in straight lines only. The section will consist, besides the axis, of a series of elements, equally spaced.

Any plane perpendicular to the axis will contain but a single element.

THE OBLIQUE HELICOID

237. Generation of the Surface. The **oblique helicoid** differs from the right helicoid in that the angle between the generatrix and the directing helix is other than a right angle. The only form of the surface which we shall consider is the following. A helix and its axis are taken as linear directrices, and the generatrix makes with the axis a constant angle other than 90°.

238. Representation of the Surface. The surface, like the right helicoid, consists of two nappes. But since the generating line is oblique to the axis, these nappes will intersect each other. On this account it is difficult to visualize more than a very limited portion of the surface.

In Fig. 375 is projected, for one turn of the directing helix, the portion of the surface bounded by this helix and its axis. That is, only one nappe is shown. The initial element is shown in V-projection at $0^v b^v$. After one complete turn of the directing helix, the element is in the parallel position $12^v c^v$. The distance $b^v c^v$ along the axis is evidently the pitch of the helix. Now if the points on the helix have been obtained by dividing the pitch into 12 equal parts, divide $b^v c^v$ also into 12 equal parts. By connecting the points of division with the corresponding points on the helix, we obtain the ele-

ments, as shown. This V-projection possesses a contour, which is drawn as the envelope of the elements.

It is evident that every point in the generating line will describe a helix. The one generated by the point f is shown in Fig. 375.

Problem 63. ***To pass a plane tangent to an oblique helicoid at a given point in the surface.***

Analysis. The plane is determined by the element which contains the given point, and by the line tangent to the helix of the surface which passes through the point (§ 155).

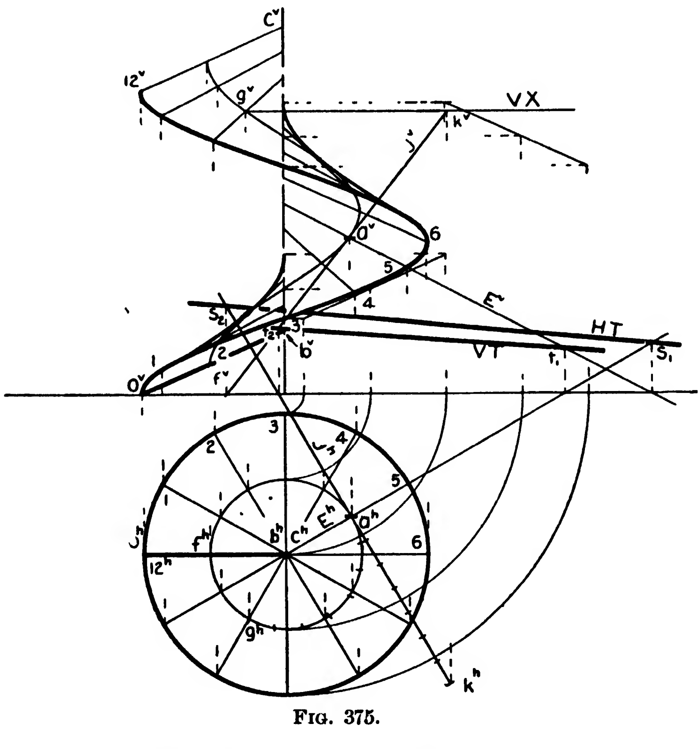

FIG. 375.

Construction (Fig 375). Let a, lying in the element E, be the given point. Through a^h, draw the H-projection, a^hf^h, of the helix of the surface passing through point a. The com-

plete V-projection of this helix is not needed; but we should obtain g^v, the point in which this helix intersects some element of the helicoid. Draw J^h tangent to the projected helix at a^h. Make the distance a^hk^h equal to the length of the arc a^hg^h. The points k and g must lie in the same horizontal plane, hence pass the horizontal plane X through point g; that is, draw VX through g^v. Project k^h to k^v in VX. Join k^v to a^v, thus obtaining J^v, the V-projection of the tangent to the helix at a. (See Prob. 60.) Pass the required tangent plane T through the lines E and J.

Problem 64. ***To find the intersection of an oblique helicoid and a plane.***

Analysis. Points are obtained by finding where elements of the surface intersect the given plane (§§ 173, 221).

Construction (Fig. 376). Let the axis of the surface be given perpendicular to H, and let Q be the cutting plane. In finding where elements of the surface intersect the plane Q, it is decidedly preferable to take through the elements auxiliary planes which are perpendicular to H. These planes will then all contain the axis of the surface, hence the lines of intersection of these planes and Q will all pass through a common point, n, in the axis. This common point is similar to that found in the construction of the intersection of a cone and plane. (See Prob. 41.)

The points on the elements 3 and 9, which lie in a profile plane, must be specially found by an auxiliary profile projection. Since the axis is a line lying in the surface, the point n, found in the axis, is one of the points in the required intersection, and should so appear when the intersection is found. In Fig. 376 the point n happens to be identical with the point in which the element 0 intersects the plane Q.

A. A Line Tangent to the Intersection.

Analysis. See §§ 178 and 221.

Construction. Let a, on the element 4, be the given point. Find the plane W, tangent to the given surface at the point a (Prob. 63). The H-trace, HW, is sufficient. The intersection

of *HW* and *HQ* gives the *H*-trace, *s*, of the required tangent line *T*, which is now drawn through *a* and *s*.

239. Forms of the Intersection. Infinite Branches. The intersection of an oblique helicoid and a plane may or may not contain infinite branches.

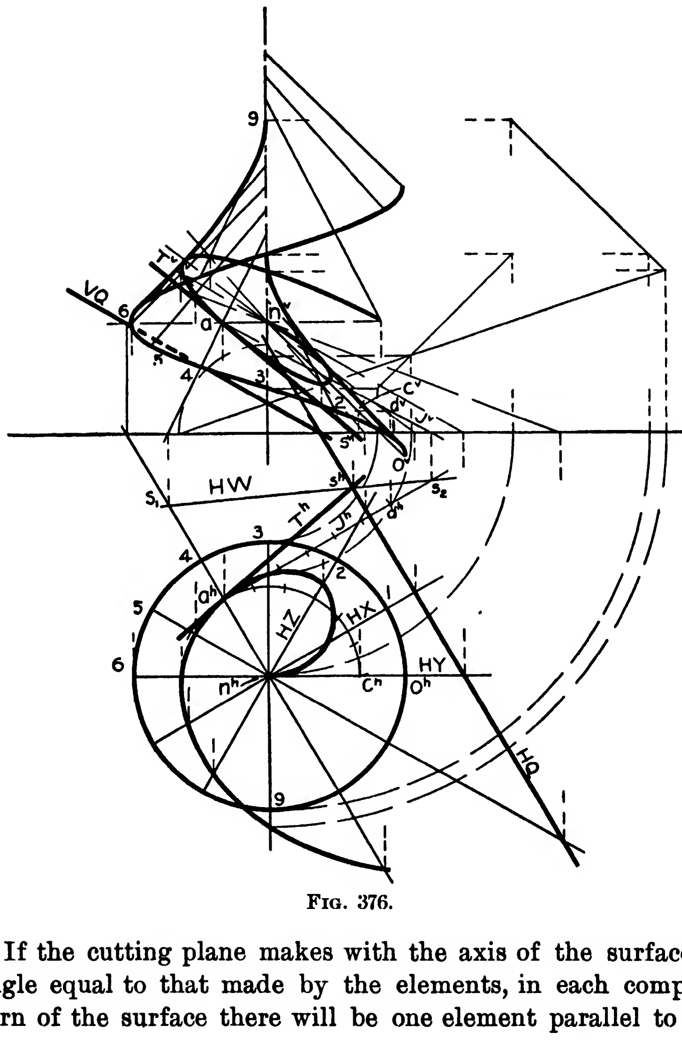

FIG. 376.

If the cutting plane makes with the axis of the surface an angle equal to that made by the elements, in each complete turn of the surface there will be one element parallel to the

plane. Therefore the curve of intersection will break into infinite branches. If the cutting plane makes with the axis an angle less than that made by the elements, there will be two elements parallel to the plane in each complete turn of the surface; hence there will be infinite branches. But if the cutting plane makes with the axis an angle greater than that made by the elements, every element of the surface will intersect the plane, and the complete intersection will be a continuous curve. This is the case shown in Fig. 376. In particular, if the cutting plane is perpendicular to the axis of the surface, the section will be a spiral of Archimedes.

THE RIGHT CONOID

240. Conoids. The general definition of a **conoid** is that it is a warped surface having one plane directrix, one rectilinear directrix, and one curve directrix, the latter of which may be either a plane or a space curve. The curve directrix, however, must be so chosen that a hyperbolic paraboloid is not formed.

If the plane directrix is at right angles to the rectilinear directrix, the surface is called a right conoid. In the right conoid, every element is perpendicular to the rectilinear directrix. The right helicoid which we have already considered is included in the definition of a right conoid.

241. Line of Striction. In any warped surface, the **line of striction** is the shortest line, straight or curved, which can be drawn on the surface, perpendicular to all the elements. In most warped surfaces, the position of this line is not obvious. But in the right conoid, the rectilinear directrix is evidently the line of striction; hence it is commonly called by that name in discussing the right conoid.

242. The Symmetric Right Conoid. A symmetric form of the right conoid, which will be the only one considered here, is obtained as follows. An ellipse (or circle) is taken as the base of the surface. A line perpendicular to the base at its middle point is the axis of the surface, and intersects the line of striction. The line of striction is taken parallel to one of the axes

(major or minor) of the ellipse which forms the base. Such a surface is shown pictorially in Fig. 377, and in three projections in Fig. 378.

243. The Complete Surface. As in any warped surface, the line generating the right conoid may be of indefinite length. It may be produced beyond the base, and beyond the line of striction. The complete surface will therefore consist of two equal nappes, separated by the line of striction. .

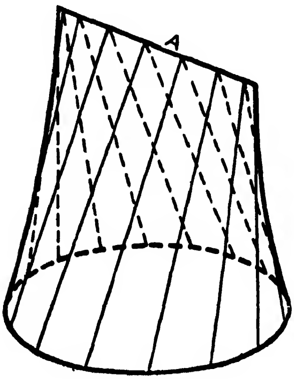

FIG. 377.

In the complete surface there will be sections, one in each nappe, equally distant from the line of striction, which will be circles. In Fig. 378, a circular section is cut in the lower nappe by the plane X. This plane is perpendicular to the axis of the surface, and $cd = ef$. Any other plane perpendicular to the axis, except the one containing the line of striction, will cut the surface in an ellipse, one axis of the ellipse being parallel to the line of striction. For convenience, the conoid is usually represented by using one of its circular sections as the base. (See Fig. 380.)

244. Planes Tangent to the Right Conoid. The conoid being placed as shown in Fig. 378, with the line of striction perpendicular to V, it is evident that tangent planes which contain the contour elements of the V-projection, and those which contain the contour elements of the P-projection, will be tangent to the surface along the entire length of these elements. These four elements furnish exceptions to the general rule, that a plane tangent to a warped surface is tangent at but a single point of the surface (§ 220 a). However, for any other element of the surface, the rule holds.

Problem 65. *To pass a plane tangent to a right conoid at a given point in the surface.*

Analysis. The required tangent plane is determined by two lines. One line is the element of the surface passing through the given point. The other is the line tangent to the ellipse (circle) obtained by passing a plane through the surface parallel to the base (§ 155).

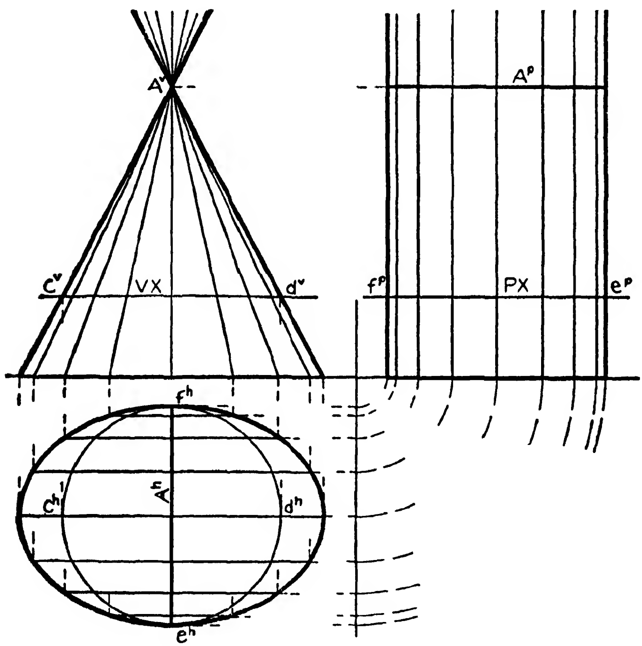

FIG. 378.

Construction (Fig. 379). Let b, lying on the element E, be the given point. Through b pass the plane X, perpendicular to the axis of the surface. This will cut from the surface an ellipse (not drawn) (§ 243). In the H-projection we have c^hd^h as the major axis of the ellipse, and b^h a point on it. We must draw J^h, tangent to the ellipse at b^h. To do this it is not necessary to draw the ellipse. On c^hd^h as diameter draw a

circle — the major auxiliary circle of the ellipse. Project b^h to this circle at b'. Draw J' tangent to the circle at b', intersecting c^hd^h produced at f. Draw J^h through f and b^h. The V-projection, J^v, of this tangent line obviously coincides with VX. Pass the required tangent plane T through the tangent line J and the element E.

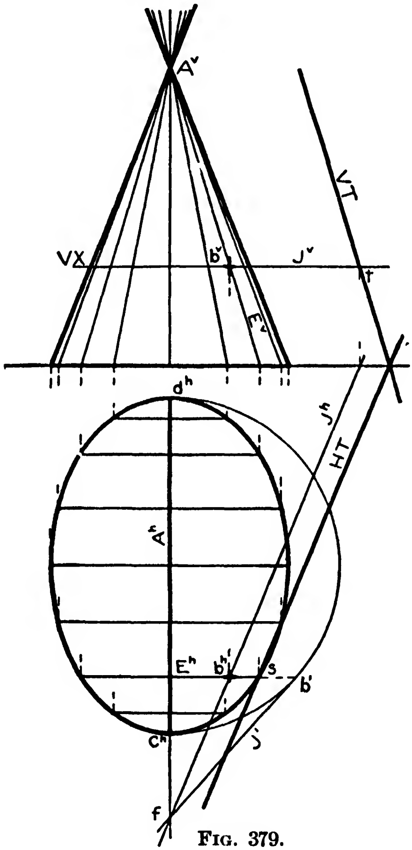

FIG. 379.

A SECOND EXAMPLE. In Fig. 381 the line of striction, A, of the conoid is oblique to V. The plane T is tangent to the surface at the point b in element E. The figure is lettered the same as Fig. 379, and the same directions will apply.

Problem 66. ***To find the intersection of a right conoid and a plane.***

Analysis. Points are obtained by finding where elements of the surface intersect the given plane (§§ 173, 221).

Construction. Example 1 (Fig. 380). The given conoid, placed with its line of striction perpendicular to *V*, is inter-

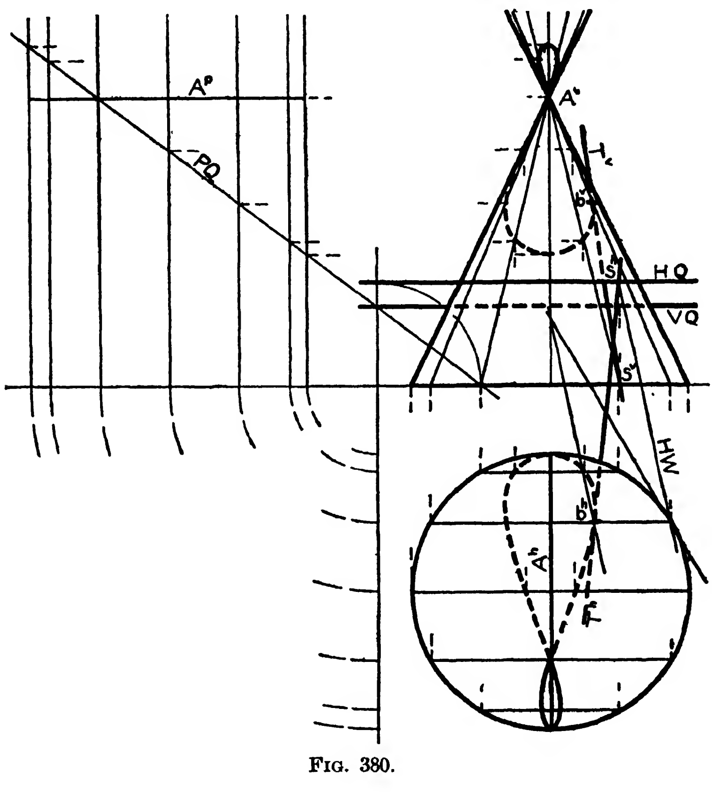

Fig. 380.

sected by the plane *Q*, parallel to the ground line. A profile projection gives the edge view, *PQ*, of the plane, and the points in the *H*- and *V*-projections are readily obtained by projecting from the *P*-projection.

In this instance, the secant plane intersects the line of striction. The intersection consists of two loops, one in each

nappe, the point on the line of striction being a double point of the curve. If we consider the conoid to be a hollow surface, that is, not inclosing a solid, in the *H*-projection we are

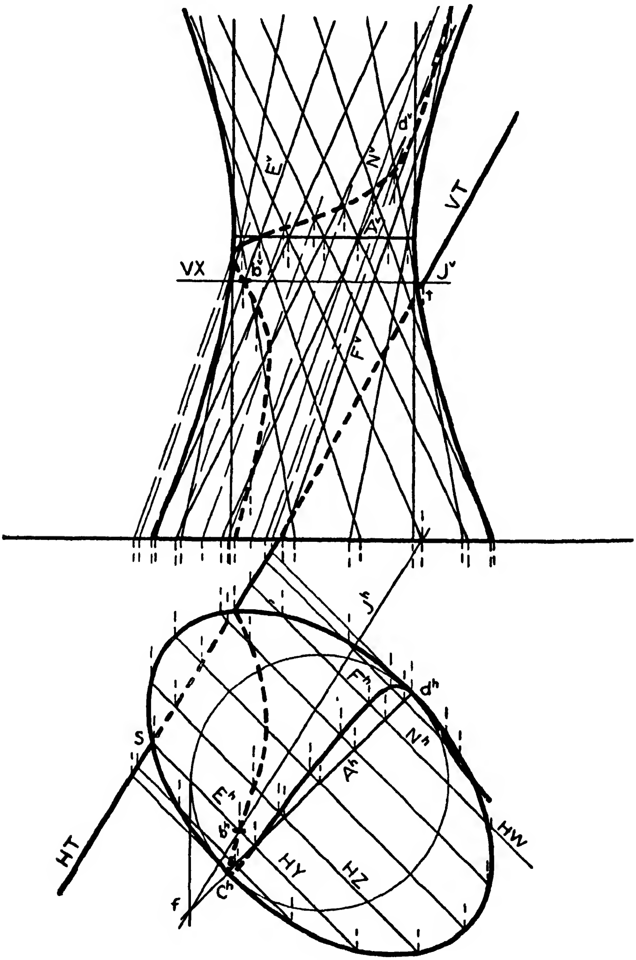

FIG. 381.

looking into the interior of the upper nappe. In this projection, then, the loop on the upper nappe is visible while the loop on the lower nappe is invisible.

A. A Line Tangent to the Intersection.

Analysis. See §§ 178 and 221.

Construction. Let b be the given point. Find HW, the H-trace of the plane tangent to the conoid at point b (Prob. 65). The intersection of HQ and HW is the H-trace, s, of the required tangent line T.

Example 2. In Fig. 381, the plane T is tangent to the given conoid at the point b. (See Prob. 65.) Let it be required to find the complete intersection of this plane with the surface.

Since the plane T contains the element E, this straight line will be a part of the required intersection. The other part will be a curve passing through the point b, found by joining the points in which various elements of the surface intersect the plane T.

Let elements be chosen as shown. Since the H-projections of these elements are parallel, the points in which they intersect T are best found by taking auxiliary planes perpendicular to H, as Y and Z. The V-traces of these planes are not needed. Since the element E lies in the plane T, the plane Y, passed through E, must intersect T in the line E. Any other auxiliary plane, as Z, parallel to Y, will therefore intersect T in a line parallel to E, and the intersection of HZ and HT is sufficient to locate the line.

B. An Asymptote to the Intersection. Let F be the element symmetrical with E; that is, F is parallel to E. The plane W, perpendicular to H, passed through F, intersects the plane T in the line N, parallel to E. But if F is parallel to E, then N is parallel to F, and the element F cannot intersect the plane T. It does not necessarily follow, from this alone, that N is an asymptote to the curve of intersection; but the construction for elements adjacent to, and on either side of F, shows it to be so.

The curved portion of the intersection is thus found to be a single curve, indefinite in extent on each nappe. It passes through the point b, intersects the line of striction at the point where element E intersects, and is asymptotic in both directions to the line N.

EXERCISES

Size of Drawings. The examples for solution which follow are designed for a working surface of $8\frac{1}{2}$ inches by $12\frac{1}{2}$ inches, or some even division thereof. This is obtained by the use of drawing paper of the so-called imperial size, about $22'' \times 30''$. Divide the sheet into quarters. Lay out a trimming line $10'' \times 14''$ for the size of the final sheet. Leave a $\frac{3}{4}''$ margin on all sides, giving the working surface $8\frac{1}{2}'' \times 12\frac{1}{2}''$.

If a cheaper paper is desired, one may use the colored detail paper specially made for pencil drawings. Good results may also be obtained by using a high-grade, heavy linen ledger paper. Ordinary note or typewriter paper is not recommended. It is either too thin, or has too soft a surface, to stand much erasure. A desirable quality for problem paper is the ability to withstand considerable erasure, and it should be selected accordingly.

Division of Working Surface. The standard working surface of $8\frac{1}{2}'' \times 12\frac{1}{2}''$ may be used entire. It may be divided by vertical lines into two or three equal parts. Or, it may be divided by one horizontal line and a sufficient number of vertical lines into six, eight, or ten equal parts. The space required for each of the following examples will be shown by giving in parentheses the number which can be placed on a standard sheet; thus, (1), (2), (3), (6), (8), or (10).

Scale of Distances. In order to avoid fractions, and the continued use of the inch mark (″), linear distances will be given in units of $\frac{1}{8}$ of an inch. Thus, 3 is $\frac{3}{8}$ of an inch, 10 is $\frac{10}{8}$ or $1\frac{1}{4}$ inches, 15 is $1\frac{7}{8}$ inches, and so on. In laying off these distances, however, do not calculate the value in inches. Use an architect's scale graduated to $\frac{1}{8}'' = 1$ foot, and lay off the units directly.

Ground Line. In each example the ground line is to be placed in the center of the allotted space, unless otherwise specified.

Origin of Coordinates. In every case, the origin of coordinates will be the intersection of the ground line with the left-hand boundary of the allotted space, whether the ground line is centrally placed or not.

Location of Points. Points will be located by three coordinates, known as the x, y, and z coordinates, respectively. Adopting the usual system of solid analytic geometry, the x coordinate represents a horizontal distance measured to the right, the y coordinate a horizontal distance measured forward, and the z coordinate a vertical distance measured upward. In other words, the x coordinate is a distance

along the ground line; the y coordinate, the distance from GL to the H-projection; the z coordinate, the distance from GL to the V-projection. Thus, point a: 3, 5, 7. Lay off 3 units from the origin along GL, and draw a vertical line. On this measure 5 units from GL, giving a^h; also 7 units from GL, giving a^v.

If one of the projections of a point is to be found in the solution of a problem, the proper letter, y or z, will be given in the data in place of the actual coordinate. Thus, point b: 6, y, 4, means that b^v, 4 units from GL, is located, but that b^h is unknown until found by solving the example.

Positive and Negative Coordinates. Since the origin of coordinates is taken always at the left of the drawing space, the x coordinate will be measured always to the right, and will be always positive. If, as in the preceding article, y is positive in a forward direction, and z in an upward direction, both y and z positive will locate a point in the first quadrant. To locate a point in the second quadrant, y must be negative; for a point in the fourth quadrant, z must be negative; while for a point in the third quadrant, both y and z must be negative.

To avoid the continued use of the plus sign $(+)$, all coordinates will be considered plus unless marked with the minus sign $(-)$.

Reversibility of the Quadrants. Suppose a number of points to be located, using the positive and negative directions of y and z as above. Let now the algebraic sign of every y and z coordinate be changed from plus to minus, and vice versa. The effect is the same as if the entire system of points is revolved 180° about GL as an axis, the points all preserving their same relative positions to each other.

The same result could also be obtained, without changing the signs of the y and z coordinates, by taking the positive direction of y to be backward, the positive direction of z downward.

Choice of Quadrants. From the preceding it follows that examples involving a series or system of points can be laid out and solved equally well in either the first or third quadrants, according to the positive direction assumed for the y and z coordinates. It will presently be shown that lines and planes, and, in general, solids, can be located by methods equally reversible. Hence, unless some particular quadrant is specified in the example, the instructor may choose between the first and third quadrant methods of location. Thus, to lay out an example in the first quadrant, the y coordinate, which locates the H-projection, is positive when measured forward from GL; the z coordinate, which locates the V-projection, is positive when measured upward from GL. To place the same example in the third quadrant, measure the y coordinate, if positive, backward from GL; measure the z coordinate, if positive, downward from GL. The third quadrant is ordinarily used

in drafting rooms where machine designing is practiced, while many architects habitually use the first quadrant for their drawings.

Angles. Angles will be measured in two semi-circular sweeps, from 0° at the right to 180° at the left, counter-clockwise above the ground line, and clockwise below the ground line. Thus, above the ground line, 15° means a line making 15° with GL, and sloping upward to the right, while 120° means 60° with GL, upward to the left. Below the ground line, 75° means 75° with GL, downward to the right, and 150° means 30° with GL, downward to the left. The use of angles between 90° and 180° thus obviates the necessity of stating the slope, which cannot be avoided if only acute angles are used.

Speaking trigonometrically, one of these sweeps of angles should be negative. In practice, however, it will be found that no ambiguity will result from the omission of the negative sign. Also, since each sweep (as drawn on the paper) is the reflection of the other by the ground line, this system of measuring angles has the same reversibility as the coordinates of points.

Location of Straight Lines. Two systems for locating the projections of straight lines will be employed. The first, by locating two points on the line. Thus, line cd: c, 3, 9, 8; d, 10, 2, 8. The line will not necessarily terminate at the located points.

The second system will be by a point on the projection, and the angle (0° to 180°) which this projection makes with GL. In every case the data for the H-projection will be given first. Thus, line A: 14, 6, 60°; 14, 4, 135°. Lay off 14 along GL; then 6 perpendicular to GL; through this point draw A^h at 60° with GL. From the same point on GL, 14 units from the origin, lay off 4 perpendicular to GL, and through this point draw A^v at 135° with GL. See the rule for laying off angles, previously given, and do not confuse 60° with 120°. If the line A is located in the first quadrant, A^h will slope forward to the right, and A^v upward to the left. The same data, if used to locate A in the third quadrant, will make A^h slope backward to the right, and A^v downward to the left.

In the above illustration, the point located on A^v is the projection of the point located on A^h. This may not always be the case. Thus, line B: 7, 0, 45°; 10, 0, 60°. This line is located by the points where its H- and V-projections, respectively, intersect GL.

If a projection of a line is parallel to GL, the coordinate along GL may (but not necessarily) be replaced by the indefinite coordinate x. Thus, line C: 9, 0, 120°; x, 6, 0°. Here, C^v is parallel to GL, and the line is parallel to H. Line D: x, 5, 0°; x, 9, 0°, is parallel to both H and V. Line E: 15, 0, 90°; 15, 8, $\odot$, is perpendicular to V, its V-projection being a point ($\odot$).

Locations of Planes. Traces of planes will be located by giving, first, the distance from the origin to the point where the traces meet on GL; second, the angle which the H-trace of the plane makes with GL; third, the angle which the V-trace makes with GL. Thus, plane Q: 12, 150°, 75°.

If a plane is parallel to the ground line, the meeting of the traces is at infinity. This will be shown by the infinity sign (∞). Since in this case each trace makes 0° with GL, the angles will be replaced by the distances from GL to the H- and V-traces in turn. Thus, plane R: ∞, 15, $-$ 9. A plane parallel to H or V has but one trace. The missing trace will be replaced by the infinity sign. Thus, plane T: ∞, ∞, 6, is parallel to, and 6 units from, H.

If the plane is not parallel to the ground line, but the intersection of its traces with GL is outside the limits of the figure, each trace will be located separately, in a manner similar to the location of the projections of lines. Thus, plane W: HW, 5, 10, 30°; VW, 25, -10, 15°. Lay off 5 along GL, then 10 perpendicular to GL. Through this point draw HW at 30°, upward or downward to the right, according as the point has been located above or below GL. Lay off 25 along GL, -10 perpendicular to GL (this locates VW on the same side of GL as HW); through this point draw VW at 15°. The traces HW and VW, if produced outside the drawing, will be found to meet GL 12.32+ units to the left of the origin.

Location of Surfaces and Solids. Surfaces are composed of points and lines, and a solid is bounded by surfaces. The location of surfaces and solids will therefore be made to depend upon the location of points and lines, according to the methods already given.

The sections (§§) referred to are those of Kenison and Bradley, Descriptive Geometry.

§§ **15 to 22.** Draw the projections of the following points, and letter them properly a^h, a^v, b^h, b^v, etc. They may be placed on a single ground line, 100 units ($12\frac{1}{2}''$) long, spaced 10 units apart.

1. Point a in the first quadrant, 8 from V and 12 from H.

2. Point b in the second quadrant, 10 from V and 4 from H.

3. Point c in the third quadrant, 14 from V and 11 from H.

4. Point d in the second quadrant, 14 from V and 6 from H.

5. Point e, in H and 14 in front of V.

6. Point f, in H and 10 behind V.

7. Point g, in V, and in H.

8. Point j, in V and 14 above H.

9. Point k, in V and 12 below H.

§§ **23–26.** Draw the plan and elevation of the following objects as described and located. They should be placed from 3 to 5 units apart. To facilitate the spacing, the extreme width of each figure, measured along GL, is given, an odd fraction of a unit being counted as a whole unit.

10. (5 units.) First quadrant. A square right prism, $5 \times 5 \times 10$. The prism stands on H, with its long edges vertical. Its back face is parallel to V, distant 2.

11. (8 units.) First quadrant. A triangular right prism. Base is an equilateral triangle, side 8, parallel to H, distant 2. Altitude of prism is 9. Back face of prism is parallel to V, distant 2.

12. (14 units.) First quadrant. A rectangular block, $6 \times 10 \times 12$. The V-projection shows as a rectangle, 6×12, with one corner in H, the long edges sloping downward to the left, 30° with H. One face of the block is parallel to V, distant 2.

13. (8 units.) Third quadrant. A triangular right pyramid. Base is an equilateral triangle, side 8; the altitude is 12. The base is in H, its front corner distant 4 from V. The left side of the base makes 45° with V.

14. (11 units.) Third quadrant. A hollow square prism. The bases are parallel to H, distant 1 and 11, respectively. Each base is a square, 8×8, with the front corner distant 2 from V, and the left front edge making 30° with V. Through the entire length of the prism is a square hole, 4×4, centrally placed, with its faces parallel to the lateral faces of the prism.

15. (12 units.) Third quadrant. A hexagonal right pyramid. The base is a regular hexagon, side 6; the altitude is 8. The axis of the pyramid is vertical. The apex of the pyramid is distant 3 from *H*. The front corner of the base is distant 2 from *V*. The front right-hand edge of the base makes 45° with *V*.

16. (12 units.) Third quadrant. A triangular right prism. The lowest lateral face of the prism is parallel to *H*, distant 8. This face is a rectangle, 7 × 10, all edges making 45° with *V*; the front corner is in *V*, the long edges are backward to the left. Each base of the prism is an isosceles triangle, base 7, altitude 5.

17. (12 units.) Third quadrant. A triangular right prism. The lateral edges are 12 long, and parallel to both *H* and *V*. The back lateral face is parallel to *V*, distant 6. This face is 6 wide, with its upper edge 3 from *H*. The bases of the prism are isosceles triangles, base 6, altitude 4.

18. (12 units.) Third quadrant. A square right pyramid, base 6 × 6, altitude 12. The base is in a plane perpendicular to *H* and making 60° with *V*, backward to the left. The front edge of the base is in *V*; the highest edge is parallel to *H*, distant 2. The vertex is to the right of the base.

19. (8 units.) Third quadrant. A cylinder of revolution, diameter 8, length 8. The axis of the cylinder is vertical, 5 behind *V*. The upper base of the cylinder is 3 below *H*. In the center of the front of the cylinder, and running its entire length, is a rectangular groove, 4 wide and 2 deep.

20. (8 units.) Third quadrant. A cone of revolution, diameter of base 8, altitude 10. The axis of the cone is vertical, distant 6 from *V*. The vertex of the cone is in *H*.

21. (8 units.) Third quadrant. A sphere, diameter 8. The center of the sphere is 6 from *V* and 8 from *H*. Convert the sphere to a hemisphere by removing the lower half.

22. (10 units.) Third quadrant. The frustum of a cone of revolution, altitude 10, diameter of lower base 10, of upper base 6. The axis of the frustum is vertical, distant 6 from *V*. The lower base is distant 13 from *H*.

§§ 27–34. Draw the projections of lines of indefinite length to pass through the quadrants named. Letter each projection, A^h, A^v, B^h, B^v, etc. Place the left-hand end of the line in the quadrant first named. State the slope of each line. Each (10).

23. Quadrants 4, 1, 2.

24. Quadrants 3, 2, 1.

25. Quadrants 1, 2, 3.

26. Quadrants 2, 1, 4.

27. Quadrants 3, 4, 1.

28. Quadrants 2, 3, 4.

29. Quadrants 3 and 1 only.

30. Quadrants 2 and 4 only.

Draw the projections of lines of indefinite length which can pass through only the quadrants named. Place the left-hand end of the line in the quadrant first named. State the plane to which each line is parallel, and the slope of the line. Each (10).

31. Quadrants 3 and 4.
32. Quadrants 3 and 2.
33. Quadrants 2 and 1.
34. Quadrants 1 and 4.
35. Quadrants 4 and 3.
36. Quadrants 1 and 2.
37. Quadrants 2 and 3.
38. Quadrants 4 and 1.
39. Quadrant 3 only.

Draw and letter the projections of the following lines. Each (10).

40. A line in the 1st quadrant, perpendicular to *H*.
41. A line in the 3d quadrant, perpendicular to *V*.
42. A line passing through the 1st and 2d quadrants, perpendicular to *V*.
43. A line in the 3d quadrant, perpendicular to *H*. Through what other quadrant can this line pass?

Problem 1.* Find the traces of the given line. Designate the *H*-trace as point *s*, the *V*-trace as point *t*, and letter both projections of each of these points. Each (10).

101. Line *A*, 10, 5, 135°; 10, 3, 30°.
102. Line *B*, 4, 2, 150°; 4, 11, 135°.
103. Line *C*, 5, 4, 150°; 5, 4, 120°.
104. Line *D*, 15, 3, 45°; 15, 10, 60°.
105. Line *E*, 10, 2, 150°; 10, 4, 60°.
106. Line *F*, 4, 5, 135°; 4, 5, 150°.
107. Line *G*, 9, − 2, 150°; 9, − 4, 60°.
108. Line *J*, 5, 3, 150°; 5, 6, 0°.
109. Line *K*, 14, 4, 0°; 14, 7, 60°.
110. Line *L*, 15, − 3, 30°; 15, − 3, 0°.

Problem 2. Find the projections of the straight line, given its *H*-trace, point *s*, and its *V*-trace, point *t*. Each (10).

201. Point *s*, 5, 7, *z*. Point *t*, 15, *y*, 4.
202. Point *s*, 16, 9, *z*. Point *t*, 4, *y*, − 5.
203. Point *s*, 15, − 6, *z*. Point *t*, 5, *y*, 2.
204. Point *s*, 5, − 7, *z*. Point *t*, 15, *y*, − 4.
205. Point *s*, 15, 5, *z*. Point *t*, 5, *y*, 2.
206. Point *s*, 6, − 4, *z*. Point *t*, 14, *y*, 4.

* Hereafter the number of each exercise consists of two parts: the last two digits form the serial count, while the digit in the hundreds place indicates the number of the problem to which the exercise is associated.

207. Point s, 14, -5, z. Point t, 6, y, -6.

208. Point s, 6, 6, z. Point t, 14, y, -2.

209. Point t, 10, y, 5. The line is parallel to H, and makes 45° with V, backward to the right.

210. Point s, 10, 5, z. The line is parallel to V, and makes 60° with H, downward to the right.

§§ **39–44**. Find the shadow of the given line, surface, or solid. The objects are all in the first quadrant.

211. (10) Line ab; a, 5, 5, 3; b, 13, 8, 6.

212. (10) Line ab; a, 4, 3, 4; b, 8, 2, 9.

213. (10) Line ab; a, 2, 5, 3; b, 10, 2, 8.

214. (10) Line ab; a, 3, 3, 8; b, 11, 6, 4.

215. (10) Line ab; a, 2, 6, 4; b, 10, 9, 4.

216. (10) Line ab; a, 3, 8, 6; b, 11, 3, 6.

217. (10) Line ab; a, 4, 8, 3; b, 4, 8, 12.

218. (10) Line ab; a, 5, 2, 10; b, 5, 8, 10.

219. (10) Line ab; a, 2, 4, 8; b, 10, 4, 8.

220. (10) Line ab; a, 5, 2, 10; b, 5, 8, 4.

221. (10) Line ab; a, 5, 10, 8; b, 5, 2, 5.

222. (10) Triangle abc; a, 1, 5, 10; b, 9, 6, 10; c, 12, 2, 4.

223. (8) A parallelogram $abcd$; a, 1, 3, 6; b, 5, 2, 10; c, 15, 9, 7; d, 11, 10, 3.

224. (8) A circle, parallel to V. Diameter 12; center at c, 8, 5, 8.

225. (8) A cylinder of revolution, diameter 12, length 15. The cylinder stands vertical, the center of the lower base at c, 7, 8, 0.

226. (8) A rectangular block. The lowest face is the rectangle $abcd$; a, 1, 6, 4; b, 3, 2, 4; c, 13, 7, 4; d, 11, 11, 4. The block is 9 high.

227. (8) A square pyramid. Base $abcd$; a, 2, 4, 7; b, 7, 4, 16; c, 16, 4, 11; d, 11, 4, 2. Vertex, point o, 9, 15, 9.

228. (8) A cone of revolution. Base parallel to V, diameter 12, center at c, 8, 6, 8. Vertex, point o, 8, 16, 8.

229. (8) A cone of revolution. Base parallel to H, diameter 12, center at c, 7, 8, 11. Vertex, point o, 7, 8, 0.

230. (8) An irregular tetrahedron. The four vertices are, points a, 1, 12, 10; b, 8, 6, 3; c, 13, 12, 3; d, 18, 3, 6.

231. (3) An irregular triangular pyramid. Base abc; a, 2, 16, 0; b, 7, 8, 0; c, 14, 21, 0. Vertex, point o, 16, 7, 15.

232. (3) The frustum, with non-parallel bases, of an irregular triangular pyramid. Lower base, abc; a, 1, 3, 6; b, 7, 11, 4; c, 13, 11, 2. Upper base, def; d, 16, 3, 13; e, 16, 8, 10; f, 22, 7, 11. The lateral edges are ad, be, and cf.

233. (3) The frustum, with non-parallel bases, of an irregular triangular pyramid. Lower base, abc; a, 1, 10, 4; b, 10, 4, 6; c, 14, 14,

0. Upper base, *def*; *d*, 13, 10, 12; *e*, 18, 8, 14; *f*, 18, 12, 9. The lateral edges are *ad*, *be*, and *cf*.

§§ **45–51.** Draw and letter the traces of planes oblique to both *H* and *V*. Each (10).

234. Plane *Q*, crossing all four quadrants, and sloping downward, forward, to the left.

235. Plane *R*, crossing all four quadrants, and sloping downward, forward, to the right.

236. Plane *S*, crossing all four quadrants, and sloping downward, backward, to the right.

237. Plane *T*, crossing all four quadrants, and sloping downward, backward, to the left.

238. Plane *W*, crossing quadrants 2, 1, and 4 only. What is the slope of this plane?

239. Plane *X*, parallel to *GL*, sloping downward and backward, and crossing quadrant 2. Through what other quadrants will this plane pass?

Draw and letter the traces of the following planes. Indicate which trace is the edge view of the plane. Each (10).

240. Plane *Q*, perpendicular to *V*, 45° with *H*.

241. Plane *R*, perpendicular to *H*, 30° with *V*.

242. Plane *S*, crossing quadrants 1 and 2 only. To what is this parallel?

243. Plane *T*, parallel to *V*, in front of *V*. Through what quadrants does this plane pass?

§ **58.** Find the true length of the given line *ab*. Find its *H*- and *V*-traces, producing the line as necessary. Point *c* is on the given line (or line produced); find the missing projection on *H* or *V*. Each (8).

244. Line *ab*; *a*, 12, 2, 11; *b*, 12, 8, 4. Point *c*, 12, 4, *z*.

245. Line *ab*; *a*, 12, 11, 11; *b*, 12, 2, 6. Point *c*, 12, − 2, *z*.

246. Line *ab*; *a*, 12, − 7, − 2; *b*, 12, − 2, − 11. Point *c*, 12, *y*, 5.

247. Line *ab*; *a*, 12, 6, − 3; *b*, 12, 11, − 14. Point *c*, 12, *y*, 6.

248. Line *ab*; *a*, 12, 10, 13; *b*, 12, − 3, 5. Point *c*, 12, *y*, 9.

249. Line *ab*; *a*, 12, − 7, − 12; *b*, 12, 6, − 3. Point *c*, 12, − 10, *z*.

250. Line *ab*; *a*, 12, − 11, − 9; *b*, 12, − 2, 3. Point *c*, 12, 5, *z*.

251. Line *ab*; *a*, 12, 3, 5; *b*, 12, 11, − 12. Point *c*, 12, *y*, − 7.

§ **59.** Find the profile projection of the given line, and its profile trace. Each (8).

252. Line *A*, 3, 0, 30°; 8, 0, 45°. Plane *P*, 16, 90°, 90°.

253. Line *B*, 12, 0, 150°; 8, 0, 120°. Plane *P*, 16, 90°, 90°.

254. Line *C*, 6, 0, 150°; 22, 0, 150°. Plane *P*, 11, 90°, 90°.

255. Line *D*, 17, 0, 120°; 17, 7, 0°. Plane *P*, 12, 90°, 90°.

§§ **60–61**. Find the profile trace of the given plane. Each (10).
256. Plane *Q*, 2, 30°, 45°. Plane *P*, 12, 90°, 90°.
257. Plane *R*, 15, 120°, 150°. Plane *P*, 10, 90°, 90°.
258. Plane *S*, 7, 60°, 135°. Plane *P*, 11, 90°, 90°.
259. Plane *T*, 7, 150°, 60°. Plane *P*, 13, 90°, 90°.
260. Plane *W*, ∞, 6, 10. Plane *P*, 10, 90°, 90°.
261. Plane *X*, ∞, − 8, 5. Plane *P*, 10, 90°, 90°.
262. Plane *Z*, ∞, − 6, ∞. Plane *P*, 10, 90°, 90°.

§§ **65–71**. Find the secondary projection of the given line, plane figure, or solid. In each example the secondary plane of projection is the plane *S*, given in the usual manner by its traces. The trace *HS* is the secondary ground line; the trace *VS* need not be drawn.

263. (10) Line *ab*; *a*, 4, 2, 9; *b*, 13, 11, 3. Plane *S*, 5, 45°, 90°.
264. (10) Line *ab*; *a*, 6, 10, − 7; *b*, 18, 3, 5. Plane *S*, 4, 150°, 90°.
265. (10) Line *ab*; *a*, 10, 10, 8; *b*, 18, 8, 4. Plane *S*, 14, 135°, 90°.
266. (10) Triangle *abc*; *a*, 3, 4, 4; *b*, 5, 6, 10; *c*, 11, 12, 6. Plane *S*, 2, 45°, 90°.
267. (10) Triangle *abc*; *a*, 3, 5, − 3; *b*, 9, 11, 10; *c*, 13, 15, 4. Plane *S*, 1, 45°, 90°.
268. (10) Plane quadrilateral *abcd*; *a*, 9, 10, 4; *b*, 13, 6, 13; *c*, 17, 2, 10; *d*, 15, 4, 7. Plane *S*, 16, 135°, 90°.
269. (8) A triangular right prism. Length 11. Base nearest *H*, triangle *abc*; *a*, 1, 2, 4; *b*, 4, 12, 4; *c*, 8, 5, 4. Plane *S*, 6, 60°, 90°.
270. (8) A right prism. Length 9. Base nearest *H*, parallelogram *abcd*; *a*, 15, 6, 2; *b*, 18, 11, 2; *c*, 24, 9, 2; *d*, 21, 4, 2. Plane *S*, 15, 120°, 90°.
271. (8) An irregular triangular pyramid. Base *abc*; *a*, 1, 4, 4; *b*, 5, 12, 2; *c*, 9, 4, 0. Vertex *o*, 14, 8, 10. Plane *S*, 11, 60°, 90°.

§ **70**. Find the edge view of the given plane. Use the plane *S* as a secondary plane of projection. Each (10).
272. Plane *Q*, 8, 45°, 30°. Plane *S*, 5, 135°, 90°.
273. Plane *Q*, 10, 150°, 135°. Plane *S*, 14, 60°, 90°.
274. Plane *Q*, 19, 135°, 150°. Plane *S*, 1, 45°, 90°.
275. Plane *Q*, 8, 30°, 120°. Plane *S*, 13, 120°, 90°.
276. Plane *Q*, 10, 60°, 150°. Plane *S*, 18, 150°, 90°.

§ **72**. Find the projections of the following prisms and pyramids. Use in each example a secondary projection of the base. Each (8).

First quadrant locations.

277. Locate *GL* 11 from top. A triangular right prism. Length 14. Plane of back base, 13, 120°, 90°. Highest corner of this base, 9, *y*, 9. The front lateral face makes 30° with *H*, and is 6 wide. The upper back lateral face makes 60° with *H*, and is 8 wide.

278. Locate *GL* 11 from top. A right prism. Length 12, bases rectangles 4 × 5. Plane of back base, *HS*, 13, 8, 30°; *VS*, not in the figure, 90°. Highest corner of this base, 13, 8, 8. The wide lateral faces make 60° with *H*, upward to the right.

279. A triangular right prism. Length 12. Plane of front base, *HS*, 6, 15, 45°; *VS*, not in the figure, 90°. Front end of highest lateral edge, 6, 15, 10. The front lateral face is vertical, and is 8 wide. The other lateral faces are each 6 wide.

280. A triangular right prism. Length 12. Plane of back base, 17, 135°, 90°. The highest lateral face is 8 wide, and is parallel to *H*. Back corner of this face, 15, 2, 10. The front lateral face is 9 wide, and the back lateral face is 7 wide.

281. A square right pyramid. Side of base 8, altitude 15. Plane of base, 17, 135°, 90°. Back corner of base, 16, 1, 6. Lower back edge of base makes 30° with *H*. Vertex to right of base.

282. Locate *GL* 11 from top. A rectangular block, 10 × 10 × 5. Back end of the lowest edge, point *a*, 8, 12, 0. This edge, length 10, lies in *H*, and makes 60° with *V*, forward to the right. The other edge of length 10 from point *a* makes 30° with *H*, upward to the right.

THIRD QUADRANT LOCATIONS.

283. Locate *GL* 23 from top. A triangular right prism. Length 16; base an equilateral triangle, side 7. Plane of front base, 17, 135°, 90°. Front corner of this base, 11, *y*, 3. The lower front lateral face makes 45° with *H*, downward, backward, to the left.

284. A triangular right prism. Length 11; base an equilateral triangle, side 8. Plane of front base, 17, 135°, 90°. Highest corner of this base, 13, 4, 4. The upper back face makes 45° with *H*.

285. A square right prism. Length 12, side of base 6. Plane of front base, 11, 60°, 90°. Front end of highest lateral edge, 13, *y*, 3. The upper front lateral face makes 60° with *H*.

286. A triangular right prism. Length 12; base an equilateral triangle, side 8. Plane of front base, 12, 60°, 90°. Front end of highest lateral edge, 13, *y*, 2. The upper lateral face makes 15° with *H*, downward and backward.

287. A triangular right pyramid. Altitude 14, base an equilateral triangle, side 8. Plane of base, 16, 120°, 90°. Highest corner of base, 11, *y*, 2. The upper front edge of the base makes 45° with *H*. Vertex to right of base.

Problem 3. Find the true length of the line connecting the given points *a* and *b*. Use the method of solution, and find the angle with *H*, or the angle with *V*, as directed by the instructor. Each (8).

301. Point *a*, 9, 8, 2. Point *b*, 16, 4, 9.

302. Point *a*, 6, 2, − 5. Point *b*, 18, 9, 7.

303. Point a, 8, $-$ 9, 1. Point b, 15, $-$ 2, 13.
304. Point a, 7, 7, 9. Point b, 19, $-$ 5, 2.
305. Point a, 6, $-$ 2, 7. Point b, 18, $-$ 9, 5.
306. Point a, 10, 9, 8. Point b, 17, 2, $-$ 4.
307. Point a, 7, $-$ 12, $-$ 1. Point b, 19, 0, $-$ 8.
308. Point a, 10, $-$ 2, 10. Point b, 17, 10, 3.
309. Point a, 9, $-$ 6, $-$ 2. Point b, 16, 6, $-$ 9.
310. Point a, 7, $-$ 2, $-$ 7. Point b, 19, 5, 5.
311. Point a, 9, 2, 7. Point b, 16, $-$ 5, $-$ 5.
312. Point a, 6, 1, $-$ 7. Point b, 18, 13, 0.
313. Point a, 6, $-$ 2, 5. Point b, 18, $-$ 9, $-$ 7.
314. Point a, 9, 6, $-$ 3. Point b, 16, $-$ 6, $-$ 10.
315. Point a, 9, 9, $-$ 2. Point b, 16, $-$ 3, 5.
316. Point a, 7, $-$ 5, 7. Point b, 19, 2, $-$ 5.

Problem 4. Find the projections of the line determined by the given conditions. Either method of solution may be used. The directions of the y and z coordinates of the located point are to be taken so as to place the located point in the quadrant named. Each (10).

401. Length 16. Line makes 30° with H, its plan makes 45° with GL.

(a) Upper end, quadrant 1, point 17, 4, 12. Slope downward, forward, to the left.

(b) Lower end, quadrant 3, point 17, 4, 12. Slope upward, backward, to the left.

402. Length 12. Line makes 45° with V, its elevation makes 30° with GL.

(a) Front end, quadrant 1, point 12, 12, 8. Slope downward, backward, to the left.

(b) Back end, quadrant 3, point 12, 12, 8. Slope upward, forward, to the left.

403. Length 12. Line makes 60° with H, its plan makes 45° with GL.

(a) Lower end, quadrant 1, point 4, 6, 3. Slope upward, forward, to the right.

(b) Upper end, quadrant 3, point 4, 6, 3. Slope downward, backward, to the right.

404. Length 14. Line makes 30° with V, its elevation makes 45° with GL.

(a) Front end, quadrant 1, point 4, 11, 7. Slope downward, backward, to the right.

(b) Back end, quadrant 3, point 4, 11, 7. Slope upward, forward, to the left.

405. Length 14. Line makes 45° with H, its plan makes 60° with GL.

(a) Upper end, quadrant 2, point 5, − 6, 12. Slope downward, forward, to the right.

(b) Lower end, quadrant 4, point 5, − 6, 12. Slope upward, backward, to the right.

406. Length 14. Line makes 60° with V, its elevation makes 45° with GL.

(a) Front end, quadrant 1, point 14, 14, 4. Slope upward, backward, to the left.

(b) Back end, quadrant 3, point 14, 14, 4. Slope downward, forward, to the left.

407. Length 14. Line makes 30° with V, its elevation makes 60° with GL.

(a) Back end, quadrant 1, point 3, 4, 6. Slope downward, forward, to the right.

(b) Front end, quadrant 3, point 3, 4, 6. Slope upward, backward, to the right.

408. Length 14. Line makes 30° with V, its elevation makes 45° with GL.

(a) Front end, quadrant 1, point 5, 4, 14. Slope downward, backward, to the right.

(b) Back end, quadrant 3, point 5, 4, 14. Slope upward, forward, to the right.

409. Length 16. Line makes 60° with H, its plan makes 30° with GL.

(a) Upper end, quadrant 1, point 6, 6, 11. Slope downward, forward, to the right.

(b) Lower end, quadrant 3, point 6, 6, 11. Slope upward, backward, to the right.

410. Length 14. Line makes 30° with H, its plan makes 45° with GL. Slope downward, forward, to the left. The middle point of the line, 12, 10, − 5.

Problem 5. Find the projections of a line 16 units long, making the given angles with H and V. The directions of the y and z coordinates of the located end of the line are to be so taken as to place the point in the quadrant named. Each (10).

501. Line makes 30° with H, 45° with V. Right-hand end, point b, 17, 2, 12.

(a) Point b in quadrant 1 is the upper back end.

(b) Point b in quadrant 3 is the lower front end.

502. Line makes 45° with H, 30° with V. Right-hand end, point b, 17, 5, 2.

(a) Point b in quadrant 1 is the lower back end.

(b) Point b in quadrant 3 is the upper front end.

503. Line makes 30° with *H*, 45° with *V*. Left-hand end, point *a*, 3, 4, 2.

(*a*) Point *a* in quadrant 1 is the upper back end.

(*b*) Point *a* in quadrant 3 is the lower front end.

504. Line makes 45° with *H*, 30° with *V*. Left-hand end, point *a*, 3, 6, − 2.

(*a*) Point *a* in quadrant 4 is the lower back end.

(*b*) Point *a* in quadrant 2 is the upper front end.

505. Line makes 30° with *H*, 30° with *V*. Left-hand end, point *a*, 3, 5, 10.

(*a*) Point *a* in quadrant 1 is the upper back end.

(*b*) Point *a* in quadrant 3 is the lower front end.

506. Line makes 30° with *H*, 45° with *V*. Left-hand end, point *a*, 3, 14, − 6.

(*a*) Point *a* in quadrant 4 is the lower front end.

(*b*) Point *a* in quadrant 2 is the upper back end.

507. Line makes 45° with *H*, 30° with *V*. Left-hand end, point *a*, 3, 6, 4.

(*a*) Point *a* in quadrant 1 is the upper front end.

(*b*) Point *a* in quadrant 3 is the lower back end.

508. Line makes 45° with *H*, 15° with *V*. Left-hand end, point *a*, 2, 6, 14.

(*a*) Point *a* in quadrant 1 is the upper back end.

(*b*) Point *a* in quadrant 3 is the lower front end.

509. Line makes 30° with *H*, 45° with *V*. Left-hand end, point *a*, 3, 8, 14.

(*a*) Point *a* in quadrant 1 is the upper front end.

(*b*) Point *a* in quadrant 3 is the lower back end.

510. Line makes 30° with *H*, 30° with *V*. Left-hand end, point *a*, 3, 6, 4.

(*a*) Point *a* in quadrant 1 is the lower front end.

(*b*) Point *a* in quadrant 3 is the upper back end.

§§ 84–86. Find the intersection of the given solid with the given plane. In each example the ground line is merely a reference line for coordinates, and not an essential part of the figure. Each (3).

511. A cone of revolution. Axis vertical. Diameter of base 18, center at *c*, 12, 14, 24. Vertex, point *o*, 12, 14, 0. Plane *Q*; *HQ*, not in the figure, 90°; *VQ*, 12, 13, 45°. Find true size of section, using an axis parallel to *VQ*, distant 13.

512. A cone of revolution. Axis vertical. Diameter of base 18, center at *c*, 10, 14, 20. Vertex, point *o*, 10, 14, 0. Plane *Q*, 0, 45°, 90°. Find true size of section.

513. A cone of revolution. Axis vertical. Diameter of base 16, center at *c*, 11, 12, 0. Vertex, point *o*, 11, 12, 20. Plane *Q*, 17, 90°, 135°. Find true size of section, using an axis parallel to *VQ*, distant 14.

514. A sphere. Diameter 16; center at *o*, 24, 11, 11. Plane *Q*, 26, 120°, 90°. Find true size of section.

515. A sphere. Diameter 18; center at *o*, 15, 15, 15. Plane *Q*, 3, 90°, 30°. Find true size of section.

516. A sphere. Diameter 18; center at *o*, 10, 16, 16. Plane *Q*; *HQ*, 10, 13, 45°; *VQ*, not in the figure, 90°. Find true size of section.

517. A torus. Axis vertical. Center at *o*, 14, 16, 24. Outer diameter, 24; inner diameter, 10. Plane *Q*; *HQ*, not in the figure, 90°; *VQ*, 1, 18, 30°. Find true size of section, using an axis parallel to *VQ*, distant 19.

518. A torus. Axis vertical. Center at *o*, 17, 14, 8. Outer diameter, 24; inner diameter, 10. Plane *Q*, ∞, 20, ∞. Also, plane *R*, ∞, 5, ∞.

519. A torus. Axis vertical. Center at *o*, 15, 14, 18. Outer diameter, 20; inner diameter, 4. Plane *Q*; *HQ*, not in the figure, 90°; *VQ*, 15, 21, 30°. Find true size of section, using an axis parallel to *VQ*, distant 18.

520. A torus. Axis vertical. Center at *o*, 21, 15, 21. Outer diameter, 22; inner diameter, 4. Plane *Q*; *HQ*, not in the figure, 90°; *VQ*, 21, 20, 150°. Find true size of section, using an axis parallel to *VQ*, distant 18.

521. A torus. Axis perpendicular to *V*. Center at *o*, 12, 20, 14. Outer diameter, 20; inner diameter, 4. Plane *Q*, 2, 90°, 45°. Find true size of section, using an axis parallel to *VQ*, distant 13.

522. A torus. Axis perpendicular to *V*. Center at *o*, 21, 21, 14. Outer diameter, 22; inner diameter, 8. Plane *Q*; *HQ*, 21, 22, 135°; *VQ*, not in the figure, 90°. Find true size of section, using an axis parallel to *HQ*, distant 16.

§§ **87–89.** Using space (6), locate the irregular triangular pyramid *oabc*, where *o* is the vertex, and *abc* the base. Intersect this pyramid by plane *Q*, and remove the piece containing the vertex, leaving an irregular frustum. Using another space (6), develop the frustum, showing all 5 faces. In locating the first line of the development, the origin of coordinates is at the lower left-hand corner of the space.

523. Points *o*, 5, 14, 16; *a*, 20, 14, 8; *b*, 16, 3, 0; *c*, 29, 3, 8. Plane *Q*, 3, 45°, 90°. Development, point *o* at 2, 9; *oa* vertical.

524. Points *o*, 7, 1, 2; *a*, 15, 15, 13; *b*, 27, 8, 13; *c*, 19, 8, 2. Plane *Q*, 17, 90°, 120°. Development, point *o* at 3, 10, *oa* vertical.

525. Points *o*, 27, 4, 16; *a*, 7, 4, 8; *b*, 14, 14, 4; *c*, 21, 14, 0. Plane *Q*, 9, 30°, 90°. Development, point *o* at 5, 11; *oa* vertical.

526. Points *o*, 6, 0, 11; *a*, 17, 11, 0; *b*, 24, 11, 7; *c*, 28, 5, 11. Plane *Q*, 23, 150°, 90°. Development, point *o* at 6, 8; *oa* vertical.

527. Points *o*, 5, 15, 0; *a*, 19, 3, 12; *b*, 25, 3, 4; *c*, 28, 15, 0. Plane *Q*, 16, 90°, 150°. Development, point *o* at 4, 10; *oa* vertical.

528. Points *o*, 7, 12, 14; *a*, 14, 4, 0; *b*, 24, 4, 5; *c*, 27, 15, 14. Plane *Q*; *HQ*, not in the figure, 90°; *VQ*, 7, 7, 30°. Development, point *o* at 7, 15; *oa* vertical.

529. Points *o*, 26, 3, 15; *a*, 7, 3, 7; *b*, 14, 14, 0; *c*, 22, 10, 0. Plane *Q*; *HQ*, not in the figure, 90°; *VQ*, 26, 5, 150°. Development, point *o* at 5, 12; *oa* vertical.

§§ **90, 91**. Construct the projections of the given object in the required position. Obtain the final result by making first the preliminary projections or constructions specified. To locate the object in the first quadrant, omit the words in parentheses. To locate in the third quadrant, substitute each word in parenthesis for the word immediately preceding it. Each (1).

530. A right prism. Length 20; base a square, side 11. The long edges slope downward, backward, to the right (left), 45° with *H*, 30° with *V*.

Locate point *c*, 22, 4, 4. Make this the lower (upper) end of line *A*, length 20, slope downward, backward, to the right (left), 45° with *H*, 30° with *V*.

Place the prism with its long edges parallel to *H*, 30° with *V*, backward to the right (left). The back (front) end of the front (back) long edge is at point *b*, 53, 14, 8. The lower front and upper back faces make 30° with *H*.

Relocate point *b*, at 88, 14, 8. Copy the elevation just found, making the projections of the long edges parallel to A^v. Construct the corresponding plan. What is the check?

531. A right prism. Length 18; base an equilateral triangle, side 12. The long edges slope upward, backward, to the left (right), 30° with H, 45° with *V*.

Locate point *c*, 24, 20, 6. Make this the lower (upper) end of line *A*, length 18, slope upward, backward, to the left (right), 30° with *H*, 45° with *V*.

Place the prism with its long edges parallel to *H*, 45° with *V*, backward to the left (right). The front (back) end of the lowest (highest) long edge is at point *b*, 62, 20, 6. The lower front (upper back) lateral face makes 45° with *H*.

Relocate point *b* at 82, 20, 6. Copy the elevation just found, making the projections of the long edges parallel to A^v. Construct the corresponding plan. What is the check?

532. A square right prism. Base 14 × 14, length 20. The long edges are perpendicular to *V*. The lowest (highest) long edge is in *H*, with its back (front) end at point *a*, 19, 6, 0. The lower (upper) left-hand lateral face makes 30° with *H*.

Relocate point *a* at 42, 6, 0. Copy the plan just found, making the projections of the long edges at 45°. Construct the corresponding elevation.

Relocate point *a* at 81, 6, 0. Copy the elevation just found, making the projections of the long edges at 30°. Construct the corresponding plan.

533. A right pyramid. Altitude 24. Base an equilateral triangle, side 18. Place the base in *V*, its lower (upper) side parallel to *H*, the right-hand corner of the base at point *a*, 29, 0, 6.

Relocate point *a* at 56, 0, 6. Place the base in plane *W*, 56, 150°, 90°, and construct the corresponding plan and elevation.

Relocate point *a* at 79, 0, 6. Revolve the elevation just found through 60°, in such a direction as to make the base visible (invisible) in plan. Construct the corresponding plan.

534. A right pyramid. Altitude 24. The base is an equilateral triangle, side 18, lying in *H*. The back (front) edge of the base is parallel to *V*, and its right-hand end is at point *b*, 29, 6, 0. Intersect this pyramid with plane *Q*, 7, 90°, 45°. Remove the piece containing the vertex, thus obtaining an irregular frustum.

Relocate point *b* at 55, 6, 0. Place the lower (upper) base of the frustum in plane *W*, 55, 90°, 150°. Draw the elevation, and construct the corresponding plan.

Relocate point *b* at 82, 6, 0. Revolve the plan just found through 45°, so as to make the lower (upper) base visible (invisible) in elevation. Construct the corresponding elevation.

Problem 6. Find the plane determined by the two given lines. Each (8).

601. Line *A*, 12, 2, 135°; 12, 6, 30°. Line *B*, 12, 2, 30°; 12, 6, 120°.
602. Line *A*, 10, 6, 150°; 10, 8, 120°. Line *B*, 10, 6, 45°; 10, 8, 135°.
603. Line *A*, 16, 0, 45°; 3, 0, 30°. Line *B*, 6, 0, 45°; 12, 0, 30°.
604. Line *A*, 10, 4, 30°; 10, 12, 60°. Line *B*, 10, 4, 150°; 10, 12, 135°.
605. Line *A*, 4, 0, 45°; 10, 0, 120°. Line *B*, 15, 0, 45°; 21, 0, 120°.
606. Line *A*, 11, 10, 135°; 11, 8, 120°. Line *B*, 11, 10, 135°; 11, 8, 150°.
607. Line *ab*; *a*, 11, 6, 2; *b*, 11, − 3, − 12. Line *C*, 11, 6, 45°; 11, 2, 135°.
608. Line *ab*; *a*, 12, 2, 5; *b*, 12, 10, − 4. Line *C*, 12, 2, 15°; 12, 5, 150°.

Special Case I. Each (8).

609. Line *A*, 16, 7, 45°; 16, 5, 0°. Line *B*, 16, 7, 60°; 16, 5, 135°.
610. Line *A*, 6, 8, 0°; 6, 5, 150°. Line *B*, 6, 8, 120°; 6, 5, 0°.

611. Line A, 11, 10, 135°; 11, − 4, 0°. Line B, 11, 10, 0°; 11, − 4, 30°.
612. Line A, 12, 4, 30°; 12, 10, 0°. Line B, 12, 4, 0°; 12, 10, 135°.
613. Line A, 4, 0, 30°; 4, 6, 0°. Line B, 4, − 9, 135°; 4, − 8, 0°.
614. Line ab; a, 14, 8, 14; b, 14, 6, 7. Line C, 14, 6, 30°; 14, 7, 0°.

SPECIAL CASE II. Each (8).

615. Line A, 12, 7, 0°; 12, 4, 0°. Line B, 12, 7, 135°; 12, 4, 30°.
616. Line A, 7, 12, 135°; 7, − 5, 150°. Line B, 7, 12, 0°; 7, − 5, 0°.
617. Line A, 8, 3, 0°; 8, − 6, 0°. Line B, 8, 3, 60°; 8, − 6, 135°.
618. Line ab; a, 12, 2, 12; b, 12, 10, 2. Line C, 12, 6, 0°; 12, 7, 0°.

SPECIAL CASE III. Each (8).

619. Line A, x, 3, 0°; x, 10, 0°. Line B, x, 8, 0°; x, 5, 0°.
620. Line A, x, 10, 0°; x, 3, 0°. Line B, x, − 7, 0°; x, − 7, 0°.
621. Line A, x, 4, 0°; x, 4, 0°. Line B, x, 12, 0°; x, − 9, 0°.
622. Line A, x, 12, 0°; x, 9, 0°. Line B, x, 5, 0°; x, − 2, 0°.

COROLLARY I. Find the plane which contains the given line and the given point. Each (8).

623. Line A, 10, 0, 135°; 4, 0, 60°. Point c, 17, 3, 5.
624. Line A, 8, 0, 135°; 5, 0, 120°. Point c, 17, − 2, 5.
625. Line A, 19, 0, 30°; 19, 5, 0°. Point c, 12, 3, − 6.
626. Line A, 6, 5, 0°; 6, 0, 30°. Point c, 11, − 3, − 9.

COROLLARY II. Find the plane which contains the three given points. Each (8).

627. Points a, 4, 8, 5; b, 13, 8, 10; c, 17, 4, 5.
628. Points a, 8, 4, 4; b, 12, 2, 10; c, 20, 8, 2.
629. Points a, 6, 7, 2; b, 10, 3, 6; c, 17, 7, 2.
630. Points a, 10, 2, 5; b, 15, 4, 3; c, 20, − 4, 11.
631. Points a, 7, − 3, 10; b, 7, 4, − 2; c, 17, 4, 2.
632. Points a, 18, 10, 10; b, 18, 2, 4; c, 8, 2, 2.

Problem 7. Find the plane which contains the first given line, and is parallel to the second given line. Each (8).

701. Line A, 14, 0, 60°; 21, 0, 120°. Line B, 4, 3, 30°; 4, 0, 45°.
702. Line A, 15, 0, 120°; 7, 0, 30°. Line B, 10, 0, 150°; 10, 0, 135°.
703. Line A, 13, 0, 60°; 8, 0, 45°. Line B, 3, 0, 45°; 10, 0, 135°.
704. Line A, 18, 0, 120°; 10, 0, 30°. Line B, 4, 7, 150°; 4, 0, 60°.
705. Line A, 8, 5, 0°; 8, 0, 45°. Line B, 18, 0, 45°; 22, 0, 120°.
706. Line A, 8, 0, 30°; 8, 0, 60°. Line B, 16, 4, 0°; 16, 0, 120°.
707. Line A, x, 9, 0°; x, − 5, 0°. Line B, 8, 0, 135°; 17, 0, 60°.
708. Line A, 10, 0, 90°; 10, 5, ⊙. Line B, 12, 0, 30°; 18, 0, 120°.

SPECIAL CASE I. Each (8).

709. Line A, 18, 0, 120°; 12, 0, 60°. Line B, 2, 0, 60°; 2, 7, 0°.
710. Line A, 18, 0, 150°; 7, 0, 30°. Line B, 10, 6, 0°; 10, 0, 60°.
711. Line A, 11, 0, 135°; 11, 6, 0°. Line B, 18, 5, 0°; 18, 0, 120°.
712. Line A, 16, 0, 135°; 9, 0, 60°. Line B, x, 10, 0°; x, 5, 0°.

SPECIAL CASE II. Each (8).

713. Line ab; a, 10, 3, − 8; b, 10, − 8, 4. Line C, 20, 0, 120°; 16, 0, 120°.
714. Line ab; a, 5, 8, 6; b, 5, 2, 3. Line C, 4, 0, 150°; 19, 0, 150°.
715. Line ab; a, 11, 3, 8; b, 11, 6, 2. Line C, 20, 0, 150°; 13, 0, 45°.
716. Line ab; a, 11, 2, 10; b, 11, 8, 4. Line C, 16, 4, 0°; 16, 0, 60°.
717. Line C, 16, 4, 0°; 16, 0, 60°. Line ab; a, 11, 2, 10; b, 11, 8, 4.
718. Line ab; a, 5, 10, 2; b, 5, 4, 8. Line C, 17, 0, 120°; 17, 4, 0°.
719. Line C, 17, 0, 120°; 17, 4, 0°. Line ab; a, 5, 10, 2; b, 5, 4, 8.

Problem 8. Find the plane which contains the given point, and is parallel to each of the given lines. Each (8).

801. Point c, 19, 5, 3. Line A, 4, 0, 45°; 9, 0, 120°. Line B, 12, 0, 120°; 5, 0, 45°.

802. Point c, 7, 4, 2. Line A, 19, 0, 120°; 19, 0, 135°. Line B, 19, 0, 150°; 19, 0, 120°.

803. Point c, 18, 7, 3. Line A, 6, 0, 60°; 6, 4, 0°. Line B, 8, 6, 0°; 8, 0, 45°.

804. Point c, 8, 2, 6. Line A, 12, 0, 45°; 14, 0, 60°. Line B, 18, 0, 60°; 18, 4, 0°.

805. Point c, 7, 3, 5. Line A, 21, 0, 135°; 12, 0, 60°. Line B, 15, 0, 135°; 22, 0, 150°.

806. Point c, 15, 6, 3. Line A, 5, 0, 90°; 5, 4, ⊙. Line B, 11, 0, 120°; 3, 0, 135°.

807. Point c, 15, 7, 8. Line ab; a, 5, 4, 10; b, 5, − 5, − 2. Line D, 5, 0, 60°; 15, 0, 45°.

808. Point c, 9, 3, 4. Line ab; a, 18, 0, 11; b, 18, − 4, − 2. Line D, 11, 0, 60°; 11, 6, 0°.

Problem 9. Find the plane which contains the given point and is parallel to the given plane. Each (8).

901. Point a, 20, 3, 6. Plane Q, 3, 30°, 45°.
902. Point a, 9, 5, 4. Plane Q, 11, 135°, 150°.
903. Point a, 15, 4, 5. Plane Q, 15, 120°, 30°.
904. Point a, 10, 4, 6. Plane Q, 17, 30°, 120°.
905. Point a, 22, − 8, 5. Plane Q, 12, 150°, 60°.
906. Point a, 16, − 9, 3. Plane Q, 14, 120°, 30°.

907. Point *a*, 10, 3, 6. Plane *Q*, 14, 45°, 90°.
908. Point *a*, 9, 3, 8. Plane *Q*, 12, 90°, 135°.

Special Case. Each (8).

909. Point *a*, 12, 4, 3. Plane *Q*, ∞, 11, 9.
910. Point *a*, 12, 7, 6. Plane Q, ∞, 11, − 4.
911. Point *a*, 12, 9, 9. Plane *Q*, contains *GL*, quadrants 1 and 3, 60° with *H*.
912. Point *a*, 12, 6, − 8. Plane *Q*, contains *GL*, quadrants 2 and 4. 60° with *V*.

§ 108. Find the plane determined by the two given lines, using a suitable auxiliary line in each example. Each (8).
913. Line *ab*; *a*, 3, 4, 5; *b*, 18, 3, 7. Line *C*, 18, 3, 135°; 18, 7, 60°.
914. Line *ab*; *a*, 7, 7, 11; *b*, 21, 9, 9. Line *C*, 7, 7, 150°; 7, 11, 120°.
915. Line *A*, 12, 0, 30°; 12, 0, 60°. Line *B*, 12, 0, 45°; 12, 0, 30°.
916. Line *A*, 7, 0, 45°; 7, 0, 30°. Line *B*, 7, 0, 120°; 7, 0, 150°.
917. Line *A*, 20, 8, 15°; 20, 12, 45°. Line *B*, 20, 8, 150°; 20, 12, 75°.
918. Line *A*, 12, 9, 45°; 12, 9, 15°. Line *B*, 12, 9, 135°; 12, 9, 135°.
919. Line *ab*; *a*, 3, 8, 8; *b*, 21, 10, 0. Line *C*, 3, 8, 150°; 3, 8, 120°.
920. Line *A*, 17, 3, 150°; 17, 8, 15°. Line *B*, 17, 3, 15°; 17, 8, 150°.

Problem 10. Find the plane which contains the given point and is perpendicular to the given line. Each (10).

1001. Point *c*, 10, 3, 3. Line *A*, 2, 0, 30°; 6, 0, 60°.
1002. Point *c*, 4, 5, 4. Line *A*, 6, 0, 60°; 4, 0, 45°.
1003. Point *c*, 10, 10, 4. Line *A*, 17, 0, 150°; 5, 0, 60°.
1004. Point *c*, 6, 7, 7. Line *A*, 12, 0, 150°; 4, 0, 60°.
1005. Point *c*, 6, 9, 4. Line *A*, 17, 0, 45°; 8, 0, 150°.
1006. Point *c*, 9, 2, − 5. Line *A*, 4, 0, 60°; 15, 0, 30°.
1007. Point *c*, 13, 5, 7. Line *A*, 3, 0, 30°; 15, 0, 135°.
1008. Point *c*, 6, 4, 5. Line *A*, 18, 0, 30°; 3, 0, 30°.
1009. Point *c*, 5, 11, 5. Line *A*, 7, 6, 0°; 7, 0, 60°.
1010. Point *c*, 15, 3, 8. Line *A*, 3, 0, 45°; 3, 6, 0°.

Special Case. Each (8).

1011. Point *c*, 17, − 6, − 5. Line *ab*; *a*, 9, 4, 10; *b*, 9, 8, 0.
1012. Point *c*, 8, 4, − 2. Line *ab*; *a*, 15, − 3, 6; *b*, 15, 8, 5.
1013. Point *c*, 12, − 3, 7. Line *ab*; *a*, 12, − 9, − 3; *b*, 12, 0, 12.
1014. Point *c*, 6, 2, 12. Line *ab*; *a*, 16, − 7, 4; *b*, 16, 7, − 4.
1015. Point *c*, 8, 5, 1. Line *ab*; *a*, 15, − 8, − 6; *b*, 15, − 2, 6.
1016. Point *c*, 12, 5, − 4. Line *ab*; *a*, 12, 2, 3; *b*, 12, 9, − 8.
1017. Point *c*, 5, − 3, 6. Line *ab*; *a*, 13, 9, 9; *b*, 13, 3, − 6.
1018. Point *c*, 20, 3, 8. Line *ab*; *a*, 11, 6, 2; *b*, 11, 2, 7.

Problem 11. Find the plane which contains the given line and is perpendicular to the given plane. Each (10).

1101. Line A, 6, 0, 60°; 2, 0, 45°. Plane Q, 9, 45°, 30°.
1102. Line A, 10, 0, 120°; 15, 0, 150°. Plane Q, 5, 60°, 45°.
1103. Line A, 12, 0, 135°; 4, 0, 30°. Plane Q, 14, 60°, 150°.
1104. Line A, 10, 0, 135°; 6, 0, 60°. Plane Q, 14, 135°, 60°.
1105. Line A, 17, 5, 0°; 17, 0, 135°. Plane Q, 15, 135°, 150°.
1106. Line A, 9, 0, 60°; 9, 4, 0°. Plane Q, 17, 135°, 120°.
1107. Line A, 9, 0, 90°; 9, 6, $\odot$. Plane Q, 15, 120°, 150°.
1108. Line A, 10, 8, 0°; 10, 7, 0°. Plane Q, ∞, 6, 10.

SPECIAL CASE I. Each (10).

1109. Line ab; a, 13, 2, 7; b, 13, 6, 2. Plane Q, 18, 120°, 135°.
1110. Line ab; a, 7, 11, 8; b, 7, 3, -5. Plane Q, 3, 45°, 30°.
1111. Line ab; a, 10, 8, -10; b, 10, 2, 12. Plane Q, 7, 60°, 30°.

SPECIAL CASE II. Each (10).

1112. Line A, 10, 6, 45°; 10, 5, 60°. Plane Q, ∞, 4, 8.
1113. Line A, 10, 8, 60°; 10, 6, 0°. Plane Q, ∞, 5, 10.
1114. Line A, 10, 4, 30°; 10, 7, 135°. Plane Q, contains GL, quadrants 2 and 4, 60° with H.

Problem 12. Find the line of intersection of the given planes. Each (10).

1201. Planes Q, 3, 60°, 45°, and R, 17, 150°, 120°.
1202. Planes Q, 4, 60°, 150°, and R, 14, 150°, 60°.
1203. Planes Q, 16, 120°, 150°, and R, ∞, 7, 8.
1204. Planes Q, 9, 150°, 60°, and R, ∞, 6, 9.
1205. Planes Q, 11, 120°, 135°, and R, ∞, 10, -4.
1206. Planes Q, 11, 150°, 135°, and R, ∞, -4, 9.
1207. Planes Q, 5, 45°, 135°, and R, ∞, 11, -5.
1208. Planes Q, 12, 150°, 60°, and R, 4, 90°, 30°.
1209. Planes Q, 7, 90°, 45°, and R, ∞, 8, 6.
1210. Planes Q, 12, 135°, 90°, and R, ∞, 5, -11.

SPECIAL CASE I. Each (10).

1211. Planes Q, 3, 30°, 60°, and R, 14, 30°, 135°.
1212. Planes Q, 10, 120°, 120°, and R, 16, 150°, 120°.
1213. Planes Q, 4, 60°, 150°, and R, 8, 60°, 135°.
1214. Planes Q, 6, 150°, 150°, and R, 13, 120°, 150°.
1215. Planes Q, 7, 120°, 30°, and R, 15, 150°, 30°.
1216. Planes Q, 4, 90°, 45°, and R, 16, 90°, 120°.
1217. Planes Q, 4, 30°, 90°, and R, 16, 135°, 90°.

Special Case II. Each (10).

1218. Planes Q, ∞, 5, 9, and R, ∞, 10, 4.

1219. Planes Q, ∞, 8, 11, and R, ∞, -9, 4.

1220. Planes Q, ∞, 9, 5, and R, ∞, -8, 11.

1221. Planes Q, ∞, 3, -10, and R, ∞, 7, 12.

1222. Planes Q, ∞, 12, -4, and R, ∞, 6, 7.

Special Case III. Each (10).

1223. Planes Q, 14, 120°, 135°, and R, ∞, 6, ∞.

1224. Planes Q, 4, 30°, 45°, and R, ∞, ∞, 6.

1225. Planes Q, 9, 45°, 150°, and R, ∞, 6, ∞.

1226. Planes Q, 7, 120°, 45°, and R, ∞, ∞, 6.

1227. Planes Q, 14, 150°, 60°, and R, ∞, 4, ∞.

Special Case IV. Each (10).

1228. Planes Q, 3, 45°, 60°, and R, 11, 150°, 45°.

1229. Planes Q, 5, 30°, 120°, and R, 12, 15°, 150°.

1230. Planes Q, 7, 120°, 150°, and R, 14, 150°, 135°.

1231. Planes Q, 6, 30°, 150°, and R, 17, 120°, 135°.

Special Case V. Each (10).

1232. Planes Q, 10, 60°, 150°, and R, 10, 135°, 30°.

1233. Planes Q, 12, 30°, 60°, and R, 12, 135°, 135°.

1234. Planes Q, 5, 30°, 30°, and R, 5, 135°, 60°.

1235. Planes Q, 5, 30°, 60°, and R, 5, 45°, 30°.

Special Case VI. In each example the plane R contains the ground line. Each (10).

1236. Planes Q, 16, 135°, 135°, and R, quadrants 1 and 3, 30° with H.

1237. Planes Q, 4, 60°, 30°, and R, quadrants 2 and 4, 60° with H.

1238. Planes Q, 18, 120°, 150°, and R, quadrants 1 and 3, 30° with H.

1239. Planes Q, 17, 150°, 120°, and R, quadrants 1 and 3, 60° with H.

1240. Planes Q, 3, 30°, 60°, and R, quadrants 2 and 4, 30° with H.

1241. Planes Q, 3, 30°, 60°, and R, quadrants 2 and 4, 45° with H.

1242. Planes Q, 5, 30°, 120°, and R, quadrants 1 and 3, 45° with H.

1243. Planes Q, 12, 135°, 60°, and R, quadrants 1 and 3, 30° with H.

1244. Planes Q, 13, 150°, 45°, and R, quadrants 1 and 3, 45° with H.

1245. Planes Q, ∞, 8, 8, and R, quadrants 2 and 4, 15° with H.

1246. Planes Q, ∞, 10, -4, and R, quadrants 1 and 3, 45° with H.

1247. Planes Q, ∞, 5, -11, and R, quadrants 1 and 3, 30° with H.

1248. Planes Q, ∞, 9, 4, and R, quadrants 2 and 4, 60° with H.

Problem 13. Find the point in which the given line intersects the given plane.

General Method. Each (8).

1301. Line A, 24, 0, 150°; 12, 0, 45°. Plane Q, 6, 30°, 60°.

1302. Line A, 15, 0, 45°; 23, 0, 135°. Plane Q, 15, 150°, 120°.

1303. Line ab; a, 12, 5, 4; b, 12, − 2, 13. Plane Q, 12, 30°, 120°.
1304. Line ab; a, 12, 3, 6; b, 12, 10, − 5. Plane Q, 12, 120°, 30°.

USUAL METHOD. Each (10).

1305. Line A, 13, 0, 30°; 8, 0, 60°. Plane Q, 13, 120°, 30°.
1306. Line A, 3, 0, 30°; 5, 0, 45°. Plane Q, ∞, 5, 11.
1307. Line A, 8, 0, 120°; 8, 4, 0°. Plane Q, 11, 150°, 60°.
1308. Line A, 9, 0, 150°; 9, − 4, 0°. Plane Q, 14, 120°, 30°.
1309. Line A, 10, 0, 135°; 10, 10, 0°. Plane Q, ∞, 4, 6.
1310. Line A, 15, 4, 0°; 15, 0, 135°. Plane Q, ∞, 8, 10.
1311. Line A, x, 5, 0°; x, 3, 0°. Plane Q, 7, 30°, 135°.
1312. Line A, x, 6, 0°; x, − 2, 0°. Plane Q, 4, 30°, 45°.
1313. Line ab; a, 10, 7, 6; b, 10, 1, 2. Plane Q, 5, 60°, 60°.
1314. Line ab; a, 10, 9, 7; b, 10, − 9, − 2. Plane Q, 3, 30°, 60°.
1315. Line ab; a, 10, 7, 6; b, 10, − 9, − 3. Plane Q, 2, 30°, 30°.
1316. Line ab; a, 10, 3, 7; b, 10, 6, 2. Plane Q, 5, 120°, 30°.
1317. Line ab; a, 10, 8, 11; b, 10, − 4, 2. Plane Q, ∞, 6, 7.
1318. Line ab; a, 10, 3, 8; b, 10, 9, 2. Plane Q, contains GL, quadrants 1 and 3, 30° with H.

SPECIAL CASE. Each (10).

1319. Line A, 4, 0, 45°; 4, 11, 150°. Plane Q, 16, 150°, 90°.
1320. Line A, 12, 0, 30°; 9, 0, 60°. Plane Q, 4, 90°, 45°.
1321. Line A, x, 10, 0°; x, 5, 0°. Plane Q, 5, 45°, 90°.
1322. Line A, 6, 0, 45°; 11, 0, 60°. Plane Q, ∞, 8, ∞.

Problem 14. Find the shortest distance from the given point to the given plane. Obtain the projections of the line, as well as the actual distance. Each (8).

1401. Point a, 10, 10, 10. Plane Q, 4, 30°, 45°.
1402. Point a, 19, 15, 4. Plane Q, 21, 135°, 150°.
1403. Point a, 20, 3, − 7. Plane Q, 7, 135°, 120°.
1404. Point a, 18, − 8, 4. Plane Q, 3, 30°, 45°.
1405. Point a, 14, 6, 12. Plane Q, 13, 120°, 30°.
1406. Point a, 19, − 4, 10. Plane Q, 10, 30°, 120°.
1407. Point a, 19, 10, − 3. Plane Q, 10, 120°, 30°.
1408. Point a, 21, − 7, 10. Plane Q, 17, 30°, 120°.
1409. Point a, 12, −5, 13. Plane Q, 4, 120°, 30°.
1410. Point a, 8, 6, 3. Plane Q, 19, 30°, 120°.

SPECIAL CASE. Each (8).

1411. Point a, 12, 11, 9. Plane Q, ∞, 8, 6.
1412. Point a, 12, 10, − 3. Plane Q, ∞, − 10, 4.
1413. Point a, 12, − 4, 9. Plane Q, ∞, 4, − 12.
1414. Point a, 12, − 2, − 5. Plane Q, ∞, 11, 6.

Problem 15. Project the given line on the given plane. Each (8).

1501. Line *ab*; *a*, 8, 1, 10; *b*, 15, 5, 3. Plane *Q*, 15, 60°, 150°.
1502. Line *ab*; *a*, 9, 7, 1; *b*, 13, 0, 8. Plane *Q*, 19, 135°, 30°.
1503. Line *ab*; *a*, 4, − 1, − 2; *b*, 14, 5, 8. Plane *Q*, 14, 60°, 150°.
1504. Line *ab*; *a*, 6, − 10, 6; *b*, 17, 9, 6. Plane *Q*, 17, 120°, 30°.
1505. Line *ab*; *a*, 9, 12, 14; *b*, 14, − 3, 11. Plane *Q*, 5, 45°, 30°.
1506. Line *ab*; *a*, 11, 6, 6; *b*, 11, − 5, − 3. Plane *Q*, 13, 60°, 45°.
1507. Line *ab*; *a*, 15, 3, 14; *b*, 15, 3, 6. Plane *Q*, 23, 150°, 135°.
1508. Line *ab*; *a*, 15, 8, 6; *b*, 15, 1, 6. Plane *Q*, 11, 30°, 120°.
1509. Line *ab*; *a*, 14, 11, 12; *b*, 21, 4, 0. Plane *Q*, ∞, 9, 7.

Special Case. Each (10).

1510. Line *A*, 5, 3, 0°; 5, 4, 45°. Plane *Q*, 5, 45°, 45°.
1511. Line *A*, 15, 5, 135°; 15, 6, 0°. Plane *Q*, 15, 135°, 120°.
1512. Line *A*, 8, 6, 45°; 8, 2, 0°. Plane *Q*, 8, 45°, 120°.
1513. Line *A*, 10, 6, 0°; 10, 6, 30°. Plane *Q*, 10, 135°, 30°.
1514. Line *A*, 8, 4, 60°; 8, 6, 45°. Plane *Q*, 8, 90°, 45°.

§§ **122, 123.** Find the intersection of the line *C* with the plane determined by the intersecting or parallel lines *A* and *B*. The ground line is merely a reference line for coordinates, and not to be used in the solution. Each (10).

1515. Line *A*, 9, 8, 150°; 9, 8, 0°. Line *B*, 9, 8, 30°; 9, 8, 60°. Line *C*, 9, 3, 60°; 9, 4, 30°.

1516. Line *A*, 10, 8, 135°; 10, 10, 0°. Line *B*, 10, 8, 45°; 10, 10, 60°. Line *C*, 10, 4, 0°; 10, 10, 135°.

1517. Line *A*, 8, 10, 30°; 8, 12, 45°. Line *B*, 8, 4, 30°; 8, 3, 45°. Line *C*, 8, 7, 45°; 8, 12, 120°.

1518. Line *A*, 5, 12, 0°; 5, 4, 135°. Line *B*, 5, 6, 0°; 5, 15, 135°. Line *C*, 5, 3, 45°; 5, 4, 0°.

1519. Line *A*, 8, 12, 0°; 8, 10, 0°. Line *B*, 8, 6, 0°; 8, 5, 0°. Line *C*, 8, 4, 60°; 8, 10, 135°.

1520. Line *A*, 8, 7, 30°; 8, 15, 120°. Line *B*, 8, 12, 30°; 8, 0, 120°. Line *C*, 8, 14, 120°; 8, 4, 30°.

1521. Line *A*, 6, 4, 60°; 6, 4, 30°. Line *B*, 6, 4, 30°; 6, 4, 60°. Line *C*, 6, 10, 150°; 6, 11, 150°.

1522. Line *A*, 8, 8, 0°; 8, 10, 0°. Line *B*, 8, 8, 30°; 8, 10, 60°. Line *C*, 8, 5, 30°; 8, 6, 30°.

Special Case. Find the intersection of the profile line *cd* with the plane of the lines *A* and *B*. Each (10).

1523. Line *A*, 15, 5, 30°; 15, 4, 135°. Line *B*, 15, 5, 150°; 15, 4, 15°. Line *cd*; *c*, 10, 9, 7; *d*, 10, 4, 0.

1524. Line *A*, 10, 0, 30°; 10, 8, 135°. Line *B*, 10, 9, 30°; 10, 4, 135°. Line *cd*; *c*, 10, 3, 10; *d*, 10, 7, 3.

§§ **124–126.** Find the intersection of the limited plane surfaces. The planes which are determined by two parallel lines are limited in width, but indefinite in length. Show visibility for the combined figure, both surfaces being opaque. The ground line is merely a reference line for coordinates. Each (6).

1525. Triangle *abc*; *a*, 8, 12, 7; *b*, 17, 1, 15; *c*, 30, 12, 0. Triangle *def*; *d*, 4, 3, 2; *e*, 22, 15, 14; *f*, 29, 9, 9.

1526. Triangle *abc*; *a*, 4, 4, 4; *b*, 18, 15, 0; *c*, 25, 4, 15. Triangle *def*; *d*, 9, 15, 12; *e*, 16, 1, 4; *f*, 30, 9, 8.

1527. Triangle *abc*; *a*, 3, 10, 4; *b*, 27, 1, 4; *c*, 30, 10, 15. Triangle *def*; *d*, 6, 15, 8; *e*, 13, 0, 15; *f*, 27, 9, 0.

1528. Triangle *abc*; *a*, 3, 14, 12; *b*, 9, 4, 0; *c*, 30, 4, 12. Plane *JK*; line *J*, 3, 0, 45°; 18, 0, 135°; line *K*, 24, 0, 45°; 31, 0, 135°.

1529. Triangle *abc*; *a*, 6, 3, 0; *b*, 12, 15, 15; *c*, 28, 3, 7. Plane *JK*; line *J*, 12, 0, 60°; 17, 0, 105°; line *K*, 18, 0, 60°; 27, 0, 105°.

1530. Triangle *abc*; *a*, 6, 10, 6; *b*, 18, 4, 0; *c*, 26, 13, 15. Plane *JK*; line *J*, 26, 9, 150°; 26, 9, 165°; line *K*, 26, 3, 150°; 26, 4, 165°.

1531. Trapezoid *abcd*; *a*, 6, 14, 6; *b*, 18, 14, 14; *c*, 27, 2, 9; *d*, 18, 2, 3. Plane *JK*; line *J*, 9, 0, 60°; *x*, 5, 0°; line *K*, 21, 0, 60°; *x*, 12, 0°.

1532. Plane *AB*; line *A*, 0, 0, 30°; 14, 0, 135°; line *B*, 9, 0, 30°; 28, 0, 135°. Plane *JK*; line *J*, 21, 0, 135°; *x*, 4, 0°; line *K*, 31, 0, 135°; *x*, 10, 0°.

1533. Plane *AB*; line *A*, 1, 0, 15°; 32, 5, 150°; line *B*, 5, 8, 15°; 23, 0, 150°. Plane *JK*; line *J*, 28, 0, 150°; 10, 0, 60°; line *K*, 32, 6, 150°; 17, 0, 60°.

1534. Plane *AB*; line *A*, *x*, 12, 0°; *x*, 4, 0°; line *B*, *x*, 5, 0°; *x*, 11, 0°. Plane *JK*; line *J*, 7, 0, 60°; 14, 0, 45°; line *K*, 19, 0, 60°; 5, 0, 45°.

§ **127.** Find the intersection of the given plane and pyramid, and show visibility. In each example the plane is limited in width but indefinite in length. Point *o* is the vertex of the pyramid. The ground line is merely a reference line for coordinates. Each (3).

1535. Pyramid *oabc*; *o*, 7, 24, 26; *a*, 15, 5, 3; *b*, 24, 22, 9; *c*, 27, 13, 20. Plane *JK*; line *J*, 0, 20, 0°; 0, 11, 30°; line *K*, 0, 14, 0°; 0, 0, 30°.

1536. Pyramid *oabcd*; *o*, 7, 13, 24; *a*, 13, 21, 4; *b*, 25, 19, 12; *c*, 31, 7, 16; *d*, 19, 2, 8. Plane *JK*; line *J*, 12, 0, 120°; 0, 9, 30°; line *K*, 29, 0, 120°; 0, 0, 30°.

1537. Pyramid *oabc*; *o*, 19, 17, 26; *a*, 3, 19, 14; *b*, 24, 24, 3; *c*, 29, 7, 14. Plane *JK*; line *J*, 1, 0, 60°; 17, 0, 120°; line *K*, 15, 0, 60°; 32, 0, 120°.

1538. Pyramid *oabc*; *o*, 29, 3, 19; *a*, 2, 13, 12; *b*, 5, 3, 4; *c*, 16, 25, 2. Plane *JK*; line *J*, 5, 0, 45°; 29, 3, 150°; line *K*, 16, 0, 45°; 29, 16, 150°.

§ **128.** Find the intersection of the given solids and show visibility. If one of the solids is a pyramid, point o is the vertex. All prisms are indefinite in length. The ground line is merely a reference line for coordinates. Each (3).

1539. Pyramid $oabc$; o, 31, 14, 23; a, 3, 3, 0; b, 15, 23, 0; c, 24, 3, 0. Prism JKL; line J, 0, 7, 15°; 0, 17, 0°; line K, 0, 12, 15°; 0, 9, 0°; line L, 0, 0, 15°; 0, 4, 0°.

1540. Pyramid $oabc$; o, 18, 26, 12; a, 3, 0, 21; b, 13, 0, 2; c, 30, 0, 17. Prism JKL; line J, 31, 5, 0°; 31, 7, 150°; line K, 31, 12, 0°; 31, 2, 150°; line L, 31, 17, 0°; 31, 16, 150°.

1541. Pyramid $oabc$; o, 4, 20, 5; a, 11, 4, 26; b, 23, 26, 25; c, 31, 8, 5. Prism JKL; line J, 4, 5, 30°; 4, 27, 165°; line K, 4, 13, 30°; 4, 15, 165°; line L, 7, 0, 30°; 4, 20, 165°.

1542. Pyramid $oabc$; o, 2, 22, 16; a, 14, 0, 3; b, 26, 0, 21; c, 32, 22, 7. Prism JKL; line J, 2, 3, 15°; 28, 0, 135°; line K, 2, 7, 15°; 32, 4, 135°; line L, 2, 17, 15°; 21, 0, 135°.

1543. Prism ABC; line A, 2, 0, 60°; 3, 0, 60°; line B, 14, 0, 60°; 10, 0, 60°; line C, 10, 0, 60°; 17, 0, 60°. Prism JKL; line J, 2, 18, 165°; 2, 5, 0°; line K, 2, 28, 165°; 2, 20, 0°; line L, 2, 12, 165°; 2, 17, 0°.

1544. Prism ABC; line A, 0, 6, 0°; 0, 26, 165°; line B, 0, 11, 0°; 0, 10, 165°; line C, 0, 24, 0°; 0, 16, 165°. Prism JKL; line J, 20, 0, 120°; 3, 0, 60°; line K, 31, 5, 120°; 16, 0, 60°; line L, 5, 10, 120°; 10, 0, 60°.

§ **130.** Find the intersection of the given prism and sphere, and show visibility. All prisms are right prisms, the long edges perpendicular to the base. Each (3).

1545. Prism, length 32, base abc; a, 5, 26, 0; b, 19, 10, 0; c, 29, 26, 0. Sphere, diameter 30, center at o, 17, 17, 16.

1546. Prism, length 30, base abc; a, 5, 12, 0; b, 22, 29, 0; c, 26, 12, 0. Sphere, diameter 26, center at o, 17, 17, 15.

1547. Prism, length 30, base $abcd$; a, 2, 5, 0; b, 2, 28, 0; c, 19, 28, 0; d, 32, 5, 0. Sphere, diameter 26, center at o, 16, 19, 15.

1548. Prism, length 26, base $abcd$; a, 4, 11, 0; b, 4, 19, 0; c, 30, 30, 0; d, 14, 2, 0. Sphere, diameter 22, center at o, 17, 17, 13.

1549. Prism, length 30, base $abcd$; a, 4, 15, 0; b, 18, 7, 0; c, 26, 21, 0; d, 12, 29, 0. Sphere, diameter 24, center at o, 18, 16, 15.

§ **131.** Find the intersection of the given cylinder of revolution and the sphere, and show visibility. The axis of each cylinder is vertical. Each (3).

1550. Cylinder, length 30, diameter 20. Center of base, point c, 13, 13, 0. Sphere, diameter 26, center at o, 19, 17, 15.

1551. Cylinder, length 30, diameter 20. Center of base, point c, 14, 13, 0. Sphere, diameter 26, center at o, 17, 17, 15.

1552. Cylinder, length 30, diameter 18. Center of base, point c, 18, 21, 0. Sphere, diameter 24, center at o, 16, 15, 15.

1553. Cylinder, length 30, diameter 18. Center of base, point c, 11, 14, 0. Sphere, diameter 26, center at o, 19, 17, 15.

§ **132**. Find the intersection of the given cylinder of revolution with the torus, and show visibility. The ground line is used only as an axis of coordinates, and should be erased before solving the example. Each (3).

1554. Cylinder, length 36, diameter 16, axis vertical. Center (of the cylinder) at c, 11, 11, 22. Torus, axis vertical, center at o, 17, 11, 18. Outer diameter, 32; inner diameter, 8.

1555. Cylinder, length 36, diameter 16, axis perpendicular to V. Center at c, 12, 11, 20. Torus, axis vertical, center at o, 17, 11, 24. Outer diameter, 32; inner diameter, 12.

1556. Cylinder, length 38, diameter 15, axis perpendicular to V. Center at c, 11, 10, 22. Torus, axis vertical, center at o, 17, 10, 22. Outer diameter, 32; inner diameter, 12.

1557. Cylinder, length 36, diameter 20, axis perpendicular to V. Center at c, 12, 12, 18. Torus, axis vertical, center at o, 17, 12, 24. Outer diameter, 32; inner diameter, 12.

1558. Cylinder, length 36, diameter 14, axis vertical. Center at c, 16, 14, 13. Torus, axis perpendicular to V, center at o, 17, 24, 13. Outer diameter, 32; inner diameter, 8.

1559. Cylinder, length 34, diameter 24, axis perpendicular to V. Center at c, 15, 14, 20. Torus, axis vertical, center at o, 17, 14, 12. Outer diameter, 32; inner diameter, 8.

1560. Cylinder, length 38, diameter 12, axis vertical. Center at c, 16, 22, 9. Torus, axis perpendicular to V, center at o, 17, 21, 9. Outer diameter, 32; inner diameter, 16.

1561. Cylinder, length 34, diameter 14, axis perpendicular to V. Center at c, 25, 10, 20. Torus, axis vertical, center at o, 15, 10, 19. Outer diameter, 28; inner diameter, 8.

Problem 16. Find the other projection of the given line, which lies in the given plane. Each (10).

1601. Plane Q, 4, 30°, 45°. Projection A^h, 16, 0, 150°.
1602. Plane Q, 16, 150°, 135°. Projection A^h, 11, 0, 135°.
1603. Plane Q, 5, 30°, 45°. Projection A^v, 15, 0, 120°.
1604. Plane Q, 9, 45°, 45°. Projection A^h, 5, 0, 30°.
1605. Plane Q, 12, 135°, 135°. Projection A^h, 5, 0, 165°.
1606. Plane Q, 3, 45°, 30°. Projection A^v, 8, 0, 45°.
1607. Plane Q, 11, 150°, 135°. Projection A^v, 15, 0, 150°.
1608. Plane Q, 13, 45°, 30°. Projection A^v, 9, 0, 60°.

1609. Plane Q, 12, 150°, 60°. Projection A^v, 4, 0, 30°.
1610. Plane Q, 4, 30°, 135°. Projection A^h, 16, 0, 135°.
1611. Plane Q, ∞, 7, 10. Projection A^v, 15, 0, 135°.
1612. Plane Q, ∞, 10, $-$ 4. Projection A^h, 13, 0, 120°.

SPECIAL CASE I. Each (10).

1623. Plane Q, 15, 135°, 150°. Projection A^h, x, 5, 0°.
1624. Plane Q, 4, 30°, 45°. Projection A^v, 11, 0, 45°.
1625. Plane Q, 5, 30°, 120°. Projection A^h, 10, 0, 30°.

SPECIAL CASE II. Each (10).

1626. Plane Q, ∞, 11, 8. Projection A^h, x, 6, 0°.
1627. Plane Q, ∞, 7, 8. Projection A^v, x, 12, 0°.
1628. Plane Q, ∞, 4, $-$ 10. Projection A^v, x, 5, 0°.

SPECIAL CASE III. Each (10).

1629. Plane Q, 10, 135°, 90°. Projection A^v, 10, 3, 150°.
1630. Plane Q, 6, 90°, 60°. Projection A^h, 14, 0, 150°.

COROLLARY I. Find the missing projection of the given point, which lies in the given plane. Each (10).

1631. Plane Q, 5, 45°, 30°. Point a, 15, y, 3.
1632. Plane Q, 5, 45°, 30°. Point a, 15, $-$ 3, z.
1633. Plane Q, 10, 30°, 60°. Point a, 10, y, 9.
1634. Plane Q, 6, 30°, 45°. Point a, 14, 6, z.
1635. Plane Q, 8, 150°, 135°. Point a, 11, 3, z.

COROLLARY II. Find the missing projection of the given line, which lies in the given plane. Each (10).

1636. Plane Q, 5, 60°, 45°. Line ab; a, 15, y, 8; b, 15, y, 3.
1637. Plane Q, 13, 135°, 120°. Line ab; a, 6, 5, z; b, 6, 9, z.
1638. Plane Q, 7, 30°, 120°. Line ab; a, 13, y, 10; b, 13, y, 4.
1639. Plane Q, 6, 120°, 30°. Line ab; a, 14, y, 7; b, 14, y, 1.
1640. Plane Q, 10, 30°, 120°. Line ab; a, 10, 4, z; b, 10, 0, z.
1641. Plane Q, 10, 135°, 135°. Projection A^h, 10, 0, 30°.
1642. Plane Q, 10, 135°, 45°. Projection A^v, 10, 0, 120°.

COROLLARY III. In each of the following planes draw a line of maximum inclination to H. Each (10).

1643. Plane Q, 15, 120°, 135°.
1644. Plane R, 5, 30°, 45°.
1645. Plane S, 5, 45°, 150°.
1646. Plane T, 15, 150°, 45°.

In each of the following planes draw a line of maximum inclination to V. Each (10).

1647. Plane Q, 5, 30°, 60°.
1648. Plane R, 15, 135°, 135°.
1649. Plane S, 5, 135°, 60°.
1650. Plane T, 15, 30°, 120°.

Problem 17. Find the perpendicular distance between the given planes. Each (10).

1701. Plane Q, 13, 150°, 135°. Plane R, 18, 150°, 135°.
1702. Plane Q, 2, 60°, 30°. Plane R, 11, 60°, 30°.
1703. Plane Q, 2, 45°, 60°. Plane R, 11, 45°, 60°.
1704. Plane Q, 4, 120°, 45°. Plane R, 12, 120°, 45°.
1705. Plane Q, 4, 30°, 120°. Plane R, 12, 30°, 120°.

Corollary. Find the perpendicular distance from the given point to the given plane. Each (10).

1706. Point a, 16, 5, 6. Plane Q, 2, 30°, 30°.
1707. Point a, 14, 2, 2. Plane Q, 4, 45°, 45°.
1708. Point a, 5, 3, 2. Plane Q, 16, 120°, 150°.
1709. Point a, 8, 6, 9. Plane Q, 18, 135°, 150°.
1710. Point a, 15, 2, 8. Plane Q, 10, 45°, 120°.

Problem 18. Find and mark the angles which the given plane makes with H and V. Each (10).

1801. Plane Q, 4, 30°, 45°.
1802. Plane Q, 16, 150°, 120°.
1803. Plane Q, 12, 150°, 60°.
1804. Plane Q, 12, 60°, 135°.
1805. Plane Q, 8, 30°, 120°.
1806. Plane Q, 8, 120°, 30°.
1807. Plane Q, 10, 60°, 120°.
1808. Plane Q, 5, 45°, 60°.

Special Case. Each (10).

1809. Plane Q, ∞, 4, 8.
1810. Plane Q, ∞, 3, $-$ 8

Problem 19. Find the other trace of the given plane. Two results in each example.

Case I. Each (10).

1901. Given VQ, 10, 135°. Plane Q makes 60° with V.
1902. Given HQ, 10, 30°. Plane Q makes 60° with H.
1903. Given VQ, 10, 60°. Plane Q makes 45° with V.
1904. Given HQ, 10, 120°. Plane Q makes 30° with H.

Case II. Each (10).

1905. Given VQ, 10, 45°. Plane Q makes 60° with H.
1906. Given HQ, 10, 30°. Plane Q makes 45° with V.
1907. Given VQ, 10, 150°. Plane Q makes 60° with H.
1908. Given HQ, 10, 135°. Plane Q makes 75° with V.

Special Case. Each (10).

1909. Given VQ, ∞, 4. Plane Q makes 60° with V.
1910. Given HQ, ∞, 6. Plane Q makes 30° with V.

Problem 20. Find the planes which make the given angles with *H* and *V*.

General Case. Take point *o* on *GL* in the center of the space. Distance from *o* to the required plane or planes is 4. Each (8).

2001. Find four non-parallel planes which make 45° with *H* and 60° with *V*.

2002. Find four non-parallel planes which make 60° with *H* and 60° with *V*.

2003. Find two planes, one sloping downward, forward, to the left, the other downward, backward, to the right, each making 60° with *H* and 45° with *V*.

2004. Find a plane sloping downward, forward, to the right, which makes 75° with *H* and 30° with *V*.

2005. Find a plane sloping downward, backward, to the left, which makes 60° with *H* and 75° with *V*.

2006. Find a plane sloping downward, forward, to the left, which makes 30° with *H* and 75° with *V*.

2007. Find a plane sloping downward, backward, to the right, which makes 75° with *H* and 60° with *V*.

2008. Find a plane sloping downward, forward, to the left, which makes 45° with *H* and 75° with *V*.

2009. Find two planes, one sloping downward, backward, to the right, the other downward, backward, to the left, each making 75° with *H* and 30° with *V*.

2010. Find two planes, one sloping downward, backward, to the left, the other downward, forward, to the right, each making 60° with *H* and 45° with *V*.

2011. Find a plane sloping downward, backward, to the right, which makes 45° with *H* and 75° with *V*.

Special Cases. Each (10).

2012. Find two non-parallel planes which make 30° with *H* and 60° with *V*.

2013. Find a plane which slopes downward to the left, and makes 60° with *H*, 90° with *V*.

2014. Find two non-parallel planes which make 90° with *H*, 60° with *V*.

2015. Find a plane which makes 90° with *H* and 90° with *V*.

2016. Find a plane which makes 0° with *H*, 90° with *V*, and passes through quadrants 1 and 2.

2017. Find a plane which makes 90° with *H*, 0° with *V*, and passes through quadrants 1 and 4.

Problem 21. The given point lies in the given plane. Find its other projection (see Problem 16). Revolve the point about *HQ* into *H*, or *VQ* into *V*, as specified. Each (8).

2101. Plane *Q*, 19, 135°, 120°. Point *a*, 11, 4, *z*. Revolve about *HQ*.
2102. Same. Revolve about *VQ*.
2103. Plane *Q*, 9, 30°, 45°. Point *a*, 15, *y*, 10. Revolve about *VQ*.
2104. Same. Revolve about *HQ*.
2105. Plane *Q*, 13, 60°, 150°. Point *a*, 7, *y*, 5. Revolve about *VQ*.
2106. Same. Revolve about *HQ*.
2107. Plane *Q*, 15, 120°, 30°. Point *a*, 10, 5, *z*. Revolve about *HQ*.
2108. Same. Revolve about *VQ*.

Special Case. Each (8).

2109. Plane *Q*, 12, 90°, 120°. Point *a*, 8, 6, *z*. Revolve about *HQ*.
2110. Plane *Q*, 12, 45°, 90°. Point *a*, 19, *y*, 4. Revolve about *VQ*.

Corollary. The given line lies in the given plane. Find its other projection, and revolve as specified. Each (8).

2111. Plane *Q*, 18, 135°, 150°. Projection A^h, 6, 0, 30°. Revolve about *VQ*.
2112. Same. Revolve about *HQ*.
2113. Plane *Q*, 12, 45°, 150°. Projection A^h, 7, 0, 30°. Revolve about *VQ*.
2114. Same. Revolve about *HQ*.
2115. Plane *Q*, 9, 30°, 120°. Projection A^v, 17, 0, 45°. Revolve about *HQ*.
2116. Same. Revolve about *VQ*.
2117. Plane *Q*, 8, 30°, 45°. Projection A^v, 8, 0, 30°. Revolve about *HQ*.
2118. Same. Revolve about *VQ*.
2119. Plane *Q*, 7, 135°, 30°. Projection A^v, 17, 0, 90°. Revolve about *HQ*.
2120. Same. Revolve about *VQ*.
2121. Plane *Q*, 19, 150°, 135°. Projection A^h, *x*, 3, 0°. Revolve about *VQ*.
2122. Same. Revolve about *HQ*.
2123. Plane *Q*, 13, 120°, 30°. Projection A^v, *x*, 3, 0°. Revolve about *VQ*.
2124. Same. Revolve about *HQ*.
2125. Plane *Q*, ∞, 5, 7. Projection A^v, *x*, 4, 0°. Revolve about *VQ*.
2126. Same. Revolve about *HQ*.
2127. Plane *Q*, ∞, 4, $-$ 10. Projection A^h, *x*, 7, 0°. Revolve about *HQ*.
2128. Same. Revolve about *VQ*.

2129. Plane Q, 11, 45°, 90°. Projection A^v, 11, 2, 30°. Revolve about HQ.

2130. Same. Revolve about VQ.

2131. Plane Q, 12, 90°, 60°. Projection A^h, 12, 4, 30°. Revolve about VQ.

2132. Same. Revolve about HQ.

Problem 22 and Corollary. Find the angle between the two given lines, and the projections of the bisector of their acute angle (more strictly, the bisector of the pair of equal and opposite angles made by the lines indefinitely produced). Each (8).

2201. Line A, 18, 7, 30°; 18, 9, 30°. Line B, 18, 7, 150°; 18, 9, 120°.
2202. Line A, 16, 7, 45°; 16, 4, 0°. Line B, 16, 7, 120°; 16, 4, 60°.
2203. Line A, 15, 6, 30°; 15, 9, 45°. Line B, 15, 6, 120°; 15, 9, 120°.
2204. Line A, 11, 5, 30°; 11, 4, 150°. Line B, 11, 5, 135°; 11, 4, 15°.
2205. Line ab; a, 11, 9, 3; b, 11, 4, 11. Line C, 11, 4, 45°; 11, 11, 135°.
2206. Line A, 13, 5, 45°; 13, 5, 15°. Line B, 13, 5, 150°; 13, 5, 45°.
2207. Line A, 11, 5, 150°; 11, 7, 0°. Line B, 11, 5, 30°; 11, 7, 120°.
2208. Line A, 10, 4, 45°; 10, 6, 150°. Line B, 10, 4, 60°; 10, 6, 120°.
2209. Line A, 10, -3, 150°; 10, 8, 45°. Line B, 10, -3, 45°; 10, 8, 150°.
2210. Line A, 15, 12, 45°; 15, 7, 30°. Line B, 15, 12, 150°; 15, 7, 135°.
2211. Line A, 12, 10, 120°; 12, 6, 30°. Line B, 12, 10, 90°; 12, 6, $\odot$.

Special Case. Find the angle between the two given lines. Each (8).

2212. Line A, 12, 5, 120°; 12, 8, 0°. Line B, 12, 5, 0°; 12, 8, 150°.
2213. Line A, 12, 6, 0°; 12, 7, 45°. Line B, 12, 6, 150°; 12, 7, 0°.

Problem 23. Find the angle between the given line and the given plane. Each (10).

2301. Line A, 17, 0, 135°; 11, 0, 105°. Plane Q, 18, 120°, 135°.
2302. Line A, 13, 0, 120°; 13, 6, 0°. Plane Q, 1, 45°, 60°.
2303. Line A, 6, 4, 0°; 6, 0, 60°. Plane Q, 5, 120°, 30°.
2304. Line A, 19, 0, 150°; 19, 5, 0°. Plane Q, 17, 120°, 135°.
2305. Line A, x, 5, 0°; x, 6, 0°. Plane Q, 3, 45°, 60°.
2306. Line ab; a, 10, 8, 10; b, 10, 3, 4. Plane Q, 12, 135°, 150°.
2307. Line A, 18, 0, 150°; 18, 0, 150°. Plane Q, 1, 45°, 90°.
2308. Line A, 5, 0, 45°; 11, 0, 120°. Plane Q, ∞, 10, -8.
2309. Line ab; a, 10, 2, 7; b, 10, 9, -6. Plane Q, ∞, 5, -10.
2310. Line A, 17, 0, 150°; 17, 6, 0°. Plane Q, 3, 30°, 90°.

Problem 24 and Corollaries. Complete the projections of the given plane figure. Find its true shape and size. Bisect one of its interior angles, as specified. Each (8).

2401. Quadrilateral *abcd*; *a*, 6, 7, 11; *b*, 14, 3, 6; *c*, 20, 12, − 6; *d*, 11, 12, *z*. Bisect the interior angle at *a*.

2402. Quadrilateral *abcd*; *a*, 5, 7, 4; *b*, 8, 12, 9; *c*, 20, 5, 2; *d*, 12, 0, *z*. Bisect the interior angle at *a*.

2403. Polygon *abcde*; *a*, 9, 5, 8; *b*, 14, 10, 13; *c*, 19, 2, 13; *d*, 17, *y*, 8; *e*, 14, *y*, 9. Bisect the interior angle at *a*.

2404. Quadrilateral *abcd*; *a*, 7, 13, 15; *b*, 2, 8, 6; *c*, 7, 3, 6; *d*, 15, 3, *z*. Bisect the interior angle at *a*.

2405. Quadrilateral *abcd*; *a*, 5, 4, 10; *b*, 10, 9, 5; *c*, 23, 2, 5; *d*, 13, − 4, *z*. Bisect the interior angle at *b*.

2406. Quadrilateral *abcd*; *a*, 10, 13, 0; *b*, 17, 6, 4; *c*, 10, 2, 11; *d*, 3, 6, *z*. Bisect the interior angle at *b*.

2407. Quadrilateral *abcd*; *a*, 5, 2, 4; *b*, 12, − 5, 11; *c*, 20, − 5, 11; *d*, 13, 7, *z*. Bisect the interior angle at *a*.

2408. Quadrilateral *abcd*; *a*, 6, 0, 12; *b*, 12, 6, 12; *c*, 21, 6, 2; *d*, 10, *y*, 4. Bisect the interior angle at *b*.

2409. Quadrilateral *abcd*; *a*, 7, 4, *z*; *b*, 14, − 3, 7; *c*, 20, − 3, − 4; *d*, 15, 6, − 4. Bisect the interior angle at *c*.

2410. Quadrilateral *abcd*; *a*, 20, 11, 8; *b*, 20, 3, 3; *c*, 10, 7, − 3; *d*, 15, 15, *z*. Bisect the interior angle at *d*.

Problem 25. Find the shortest (perpendicular) distance from the given point to the given line. Find also the projections of the perpendicular.

General Solution. Each (10).

2501. Line *A*, 15, 0, 150°; 5, 0, 60°. Point *c*, 13, 6, 4.
2502. Line *A*, 15, 0, 135°; 1, 0, 30°. Point *c*, 7, 2, 8.
2503. Line *A*, 10, 0, 120°; 3, 0, 45°. Point *c*, 14, 3, 3.
2504. Line *A*, 10, 0, 150°; 17, 0, 150°. Point *c*, 4, 8, 5.
2505. Line *A*, 18, 0, 150°; 15, 0, 135°. Point *c*, 5, 12, 4.
2506. Line *A*, 5, 0, 135°; 5, 12, 150°. Point *c*, 5, 4, 10.
2507. Line *A*, 15, 6, 150°, 15, 0, 120°. Point *c*, 6, − 2, 6.
2508. Line *A*, 12, 0, 135°; 16, 0, 60°. Point *c*, 7, − 3, 8.
2509. Line *A*, 10, 0, 150°; 10, 0, 30°. Point *c*, 4, 8, 4.
2510. Line *A*, 2, 0, 45°; 2, 0, 45°. Point *c*, 11, 2, 9.

Special Case I. Each (10).

2511. Line *A*, 11, 4, 0°; 11, 0, 120°. Point *c*, 12, 12, 9.
2512. Line *A*, 1, 0, 30°; 1, 6, 0°. Point *c*, 12, − 2, − 3.
2513. Line *A*, 5, 10, 0°; 5, 0, 45°. Point *c*, 15, 4, 0.
2514. Line *A*, *x*, 7, 0°; *x*, 9, 0°. Point *c*, 10, 3, 4.
2515. Line *A*, 15, 8, ⊙; 15, 0, 90°. Point *c*, 5, 7, − 2.

Special Case II. Each (8).

2516. Line *ab*; *a*, 13, 2, 12; *b*, 13, 8, 4. Point *c*, 3, 10, 11.
2517. Line *ab*; *a*, 10, − 3, 8; *b*, 10, 9, − 3. Point *c*, 21, − 6, − 4.
2518. Line *ab*; *a*, 10, 7, 6; *b*, 10, − 3, − 9. Point *c*, 20, − 8, 4.

Problem 26. Find the shortest distance between the two given lines, and the projections of their common perpendicular.

General Method. Pass the auxiliary plane through the line first given. Each (8).

2601. Line *A*, 22, 0, 135°; 10, 0, 30°. Line *B*, 2, 5, 30°; 24, 0, 135°.
2602. Line *A*, 23, 0, 150°; 17, 0, 150°. Line *B*, 9, 0, 60°; 23, 0, 120°.
2603. Line *A*, 4, 0, 30°; 17, 0, 135°. Line *B*, 23, 0, 135°; 23, 6, 150°.
2604. Line *A*, 24, 0, 135°; 24, 0, 150°. Line *B*, 2, 0, 30°; 2, 0, 60°.
2605. Line *A*, 22, 0, 120°; 15, 0, 60°. Line *B*, 0, 0, 45°; 1, 4, 30°.
2606. Line *A*, 2, 0, 60°; 2, 0, 60°. Line *B*, 13, 0, 60°; 22, 0, 120°.
2607. Line *ac*; *a*, 17, 7, 2; *c*, 17, 3, 9. Line *B*, 0, 9, 0°; 0, 2, 45°.
2608. Line *ac*; *a*, 13, 3, 7; *c*, 13, 7, 3. Line *B*, 0, 10, 30°; 0, 22, 30°.

Special Case I. Each (6).

2609. Line *A*, 14, 0, 30°; 14, 12, 0°. Line *B*, 31, 0, 135°; 21, 0, 45°.
2610. Line *A*, 27, 0, 45°; 27, 6, 0°. Line *B*, 20, 8, 0°; 20, 0, 150°.
2611. Line *A*, 19, 8, 0°; 19, 0, 150°. Line *B*, 4, 0, 150°; 4, 4, 0°.
2612. Line *ac*; *a*, 25, 5, 14; *c*, 25, 15, 4. Line *B*, 22, 0, 45°; 22, 2, 0°.

Special Case II. Each (6).

2613. Line *ab*; *a*, 11, 3, 8; *b*, 11, 8, 1. Line *cd*; *c*, 23, − 7, − 5; *d*, 23, 2, 4.
2614. Line *ab*; *a*, 11, 8, 6; *b*, 11, − 2, − 4. Line *cd*; *c*, 22, − 5, 9; *d*, 22, 9, 1.

Problem 27. Find the angle between the two given planes. Each (10).

2701. Plane *Q*, 2, 45°, 60°. Plane *R*, 16, 120°, 150°.
2702. Plane *Q*, 8, 30°, 120°. Plane *R*, 18, 135°, 150°.
2703. Plane *Q*, 7, 30°, 120°. Plane *R*, 15, 60°, 150°.
2704. Plane *Q*, ∞, 6, 6. Plane *R*, 6, 120°, 30°.
2705. Plane *Q*, ∞, 4, − 7. Plane *R*, 6, 120°, 150°.
2706. Plane *Q*, 2, 45°, 60°. Plane *R*, 14, 135°, 120°.
2707. Plane *Q*, 5, 30°, 120°. Plane *R*, 5, 60°, 60°.
2708. Plane *Q*, 3, 45°, 30°. Plane *R*, contains *GL*, quadrants 1 and 3, 30° with *H*.
2709. Plane *Q*, ∞, 8, 3. Plane *R*, ∞, − 8, 6.
2710. Plane *Q*, ∞, 9, 5. Plane *R*, contains *GL*, quadrants 2 and 4, 60° with *V*.

SPECIAL CASE I. Each (10).

2711. Plane Q, 3, 60°, 45°. Plane R, 13, 60°, 120°.
2712. Plane Q, 12, 135°, 120°. Plane R, 19, 135°, 150°.

SPECIAL CASE II. Each (10).

2713. Plane Q, 3, 30°, 45°. Plane R, ∞, ∞, 5.
2714. Plane Q, 17, 135°, 150°. Plane R, ∞, 6, ∞.

Problem 28 and Corollary. Find the line of intersection, and the angle, between the given planes. Take the slopes of the planes such as to bring the line of intersection within the given angle between the traces, and above or below H as desired. Each (6).

2801. Angle between HQ and HR is 60°. Plane Q makes 30° with H. Plane R makes 45° with H.

2802. Angle between HQ and HR is 90°. Plane Q makes 30° with H. Plane R makes 60° with H.

2803. Angle between HQ and HR is 45°. Plane Q makes 45° with H. Plane R makes 30° with H.

2804. Angle between HQ and HR is 60°. Plane Q makes 45° with H. Plane R slopes 3 horizontal to 2 vertical.

2805. Angle between HQ and HR is 75°. Plane Q slopes 2 horizontal to 1 vertical. Plane R slopes 3 horizontal to 2 vertical.

2806. Angle between HQ and HR is 120°. Plane Q makes 60° with H. Plane R makes 45° with H.

2807. Angle between HQ and HR is 105°. Plane Q makes 30° with H. Plane R slopes 3 horizontal to 1 vertical.

NOTE. To insure accuracy, a slope such as 2 to 1 should not be laid off with the standard $\frac{1}{8}$ inch units, but in some multiple, as 8 to 4, or 10 to 5. A slope of 2 horizontal to 1 vertical is equal to an angle of 26° 34′ with H.

Problem 29. Find in the given plane the projections of the point whose revolved position is given. Each (8).

2901. Plane Q, 12, 45°, 45°. Revolved point in H, 14, 9. Revolve back about HQ. Find 2 results.

2902. Plane Q, 12, 150°, 60°. Revolved point in H, 3, $-$ 2. Revolve back about HQ. Find 2 results.

2903. Plane Q, 10, 135°, 120°. Revolved point in H, 10, 10. Revolve back about HQ. Find 2 results.

2904. Plane Q, 14, 60°, 150°. Revolved point in H, 22, 2. Revolve back about HQ. Find 2 results.

2905. Plane Q, 15, 120°, 120°. Revolved point in V, 20, 8. Revolve back about VQ. Find one result, by revolving through more than 90°.

2906. Plane Q, 7, 60°, 30°. Revolved point in V, 7, 10. Revolve back about VQ. Find 2 results.

2907. Plane Q, 10, 150°, 45°. Revolved point in V, 19, − 2. Revolve back about VQ. Find one result, by revolving through more than 90°.

2908. Plane Q, 8, 60°, 150°. Revolved point in V, 7, 7. Revolve back about VQ. Find 2 results.

Corollary. Find in the given plane the projections of the plane figure whose revolved position is given. In each example find but one result, by revolving through more than 90°. Each (8).

2909. Plane Q, 5, 45°, 60°. Revolved figure in V; a, 1, 12; b, 5, 5; c, 9, 12. Revolve back about VQ.

2910. Plane Q, 19, 135°, 120°. Revolved figure in V; a, 15, 7; b, 18, 13; c, 24, 10; d, 21, 4. Revolve back about VQ.

2911. Plane Q, 15, 135°, 150°. Revolved figure in H; a, 9, 10; b, 14, 9; c, 15, 14; d, 10, 15. Revolve back about HQ.

2912. Plane Q, 15, 135°, 135°. Revolved figure in V; a, 16, 12; b, 16, 5; c, 5, 5; d, 5, 12. Revolve back about VQ.

2913. Plane Q, 12, 45°, 30°. Revolved figure in H; a, 9, 9; b, 12, 4; c, 18, 10. Revolve back about HQ.

2914. Plane Q, 15, 135°, 150°. Revolved figure in H; a, 9, 14; b, 14, 6; c, 19, 14. Revolve back about HQ.

2915. Plane Q, 10, 45°, 30°. Revolved figure in H; a, 7, 11; b, 11, 6; c, 19, 9; d, 15, 14. Revolve back about HQ.

2916. Plane Q, ∞, 8, 10. Revolved figure in H; a, 3, 10; b, 6, 15; c, 12, 12. Revolve back about HQ.

2917. Plane Q, 4, 60°, 45°. Revolved figure in V; a, 1, 7; b, 7, 15; c, 7, 7. Revolve back about VQ.

2918. Plane Q, 13, 30°, 120°. Revolved figure in V; a, 1, 11; b, 1, 2; c, 10, 2; d, 10, 11. Revolve back about VQ.

Problem 30. The given point lies in the given plane. Find the projections of a line perpendicular to the plane, as specified. Each (8).

For locations in the first quadrant, omit the word in parentheses. For locations in the third quadrant, substitute the word in parentheses for the one immediately preceding.

3001. Point a, 22, y, 4, is the lower (upper) end of a line, length 12, perpendicular to plane Q, 14, 30°, 45°.

3002. Point a, 21, 5, z, is the lower (upper) end of a line, length 12, perpendicular to plane Q, 13, 45°, 60°.

3003. Point a, 3, 0, z, is the lower (upper) end of a line, length 15, perpendicular to plane Q, 9, 120°, 135°.

3004. Point a, 5, y, 10, is the upper (lower) end of a line, length 12, perpendicular to plane Q, 9, 150°, 60°.

3005. Point a, 18, y, 0, is the lower (upper) end of a line, length 12, perpendicular to plane Q, 10, 30°, 30°.

3006. Point a, 6, 4, z, is the lower (upper) end of a line, length 12, perpendicular to plane Q, 14, 135°, 150°.

3007. Point a, 10, y, 6, is the upper (lower) end of a line, length 12, perpendicular to plane Q, 8, 120°, 45°.

3008. Point a, 12, 4, z, is the lower (upper) end of a line, length 12, perpendicular to plane Q, ∞, 6, 4.

3009. Point a, 12, 0, 0, is the middle point of a line, length 16, perpendicular to plane Q, 12, 150°, 45°.

§ 149. Draw the projections of a right prism or pyramid located as specified. To keep the figure within the allotted space, the plane of the base of the solid is located by its intersection with GL, but the directions of its traces are, in general, to be determined.

FIRST QUADRANT LOCATIONS. Each (2).

3010. A pyramid, altitude 18. Its axis makes 45° with H 30° with V, and slopes downward, forward, to the left. The vertex is higher than the base. The base lies in plane Q, 25, $y°$, $z°$. The base is a rectangle; its upper long edge is the line ab; a, x, 4, 7; b, x, 12, 9. The short sides of the base are 6 long.

3011. A prism, length 16. Its axis makes 15° with H, 45° with V, and slopes downward, forward, to the left. The lower base lies in plane Q, 16, $y°$, $z°$. The base is a rectangle, of which three corners are points a, x, 5, 12; b, x, 8, 7; c, x, 12, 12.

3012. A prism, length 16. Its axis makes 30° with H, 30° with V, and slopes downward, backward, to the right. The lower base lies in plane Q, 25, $y°$, $z°$. The base is an equilateral triangle. The lowest edge of the base is the line ab; a, x, 10, 0; b, x, 16, 6.

3013. A prism, length 14. Its lower base lies in plane Q, 25, 45°, 30°. The base is a square. The lowest edge of the base is the line ab; a, x, 5, 4; b, x, 8, 6.

3014. A pyramid, length 15. Its axis makes 45° with H, 30° with V, and slopes downward, backward, to the left. The vertex is higher than the base. The base lies in plane Q, 32, $y°$, $z°$. The base is a regular pentagon, inscribed in a circle of diameter 12. Lowest corner of the base, point a, x, 9, 2. The highest edge of the base is parallel to H.

3015. A prism, length 15. Its axis slopes downward, backward, to the left, and makes 30° with H. The lower base lies in plane Q, 25, 135°, $z°$. The base is a regular hexagon; one corner is point a, x, 3, 16. The diagonally opposite corner is point d, x, 11, 3.

THIRD QUADRANT LOCATIONS. Each (2).

3016. A pyramid, altitude 14. Its axis makes 15° with H, 45° with V, and slopes downward, forward, to the left. The vertex is higher than the base. The base lies in plane Q, 15, $y°$, $z°$. The base is an isosceles triangle, sides 10, 10, 12. The side 12 is the highest; its front end, point a, x, 7, 8. Its back end, point b, x, y, 11.

3017. A prism, length 16. Its axis makes 30° with H, 45° with V, and slopes downward, forward, to the right. The lower base lies in plane Q, 30, $y°$, $z°$. The base is an equilateral triangle. The highest edge of the base is the line ab; a, x, 3, 12; b, x, 5, 7.

3018. A pyramid, altitude 14. Its axis makes 45° with H, 30° with V, and slopes downward, backward, to the right. The vertex is lower than the base. The base lies in plane Q, 25, $y°$, $z°$. The base is a square; its lower left side is the line ab; a, x, 11, 7; b, x, 4, 7.

3019. A prism, length 14. Its axis makes 45° with H, 30° with V, and slopes downward, forward, to the left. The lower base lies in plane Q, 18, $y°$, $z°$. The base is an equilateral triangle, side 10. The front edge of the base is the line ab; a, x, 6, 6; b, x, 6, z.

3020. A prism, length 14. Its axis makes 30° with H, 45° with V, and slopes downward, backward, to the right. The upper base lies in plane Q, 24, $y°$, $z°$. The base is a rectangle. Its upper long edge is the line ab; a, x, 5, 8; b, x, 2, 5. The narrow sides of the prism are 7 wide.

3021. A prism, length 14. Its upper base lies in plane Q, 25, 135°, 150°. The base is a square. The highest edge of the base is the line ab; a, x, 8, 6; b, x, 5, 4.

Problem 31. The given projection is that of two points on the surface of a cone. Find the other projections of these points, and pass a tangent plane at each point.

CASE I. The base of the cone is a circle lying in either H or V according to the location given for its center.

3101. (6) Base, diameter 12, center at c, 16, 0, 7. Vertex, point o, 30, 12, 7. Points a and b, 20, 5, z.

3102. (6) Base, diameter 10, center at c, 10, 0, 8. Vertex, point o, 25, 12, 9. Points a and b, 19, y, 10.

3103. (6) Base, diameter 12, center at c, 15, 8, 0. Vertex, point o, 31, 12, 15. Points a and b, 21, y, 6.

3104. (3) Base, diameter 14, center at c, 23, 0, 16. Vertex, point o, 7, 14, 4. Points a and b, 15, y, 11.

3105. (3) Base, diameter 12, center at c, 11, 16, 0. Vertex, point o, 30, 10, 16. Points a and b, 17, 12, z.

CASE II. The base of the cone is a circle lying in *P*.

3106. (6) Base, diameter 10, center at *c*, 17, 8, 7. Vertex, point *o*, 5, 8, 7. Points *a* and *b*, 14, *y*, 5.

3107. (6) Base, diameter 10, center at *c*, 15, 8, 8. Vertex, point *o*, 27, 6, 5. Points *a* and *b*, 19, *y*, 5.

CASE III. The base of the cone is in a plane perpendicular to *H* or *V*. Each (3).

3108. Base, center at *c*, 14, 9, 6; plan, a circle, diameter 12; elevation, a straight line, slope 150°. Vertex, point *o*, 25, 4, 23. Points *a* and *b*, 15, *y*, 12.

3109. Base, center at *c*, 20, 11, 10; plan, a straight line, slope 135°; elevation, a circle, diameter 12. Vertex, point *o*, 6, 5, 5. Points *a* and *b*, 13, *y*, 6.

3110. Base, center at *c*, 15, 22, 14; plan, a straight line, slope 60°; elevation, a circle, diameter 12. Vertex, point *o*, 31, 4, 6. Points *a* and *b*, 18, 11, *z*.

3111. Base, center at *c*, 22, 14, 4; plan, a circle, diameter 12; elevation, a straight line, slope 30°. Vertex, point *o*, 4, 14, 18. Points *a* and *b*, 14, 16, *z*.

3112. Base, center at *c*, 20, 10, 11; plan, a circle, diameter 12; elevation, a straight line, slope 135°. Vertex, point *o*, 2, 24, 0. Points *a* and *b*, 17, *y*, 3.

Problem 32. Pass planes tangent to the given cone through the given point. Two tangent planes in each example.

CASE I. The base of the cone is a circle lying in either *H* or *V* according to the location given for its center.

3201. (6) Base, diameter 10, center at *c*, 24, 6, 0. Vertex, point *o*, 5, 13, 12. Point *a*, 9, 9, 6.

3202. (6) Same cone as in the preceding example. Point *a*, 9, 9, 4.

3203. (6) Base, diameter 12, center at *c*, 13, 10, 0. Vertex, point *o*, 29, 10, 14. Point *a*, 20, 4, 9.

3204. (3) Base, diameter 12, center at *c*, 11, 0, 14. Vertex, point *o*, 25, 12, 16. Point *a*, 19, 25, 5.

3205. (3) Base, diameter 12, center at *c*, 18, 0, 14. Vertex, point *o*, 2, 16, 10. Point *a*, 14, 16, 3.

3206. (3) Base, diameter 14, center at *c*, 15, 10, 0. Vertex, point *o*, 30, 16, 16. Point *a*, 12, 12, 5.

CASE II. The base of the cone is in a plane perpendicular to H or V. Each (3).

3207. Base, center at c, 12, 10, 14; plan, a straight line, slope 45°; elevation, a circle, diameter 12. Vertex, point o, 26, 4, 5. Point a, 8, 18, 5.

3208. Base, center at c, 22, 10, 4; plan, a circle, diameter 10; elevation, a straight line, slope 30°. Vertex, point o, 5, 6, 8. Point a, 8, 11, 3.

3209. Base, center at c, 25, 6, 5; plan, a straight line, slope 135°; elevation, a circle, diameter 10. Vertex, point o, 10, 4, 8. Point a, 19, 20, 17.

3210. Base, center at c, 18, 4, 8; plan, a straight line, slope 30°; elevation, a circle, diameter 12. Vertex, point o, 4, 18, 8. Point a, 14, 8, 16.

3211. Base, center at c, 7, 8, 9; plan, a straight line, slope 30°; elevation, a circle, diameter 12. Vertex, point o, 25, 2, 12. Point a, 20, 10, 21.

Problem 33. Pass planes tangent to the given cone parallel to the given line. Two tangent planes in each example.

CASE I. The base of the cone is a circle lying in either H or V, according to the location given for its center. Each (3).

3301. Base, diameter 12, center at c, 11, 14, 0. Vertex, point o, 24, 4, 13. Line A, 12, 26, 135°; 12, 20, 30°.

3302. Base, diameter 14, center at c, 9, 0, 16. Vertex, point o, 24, 12, 4. Line A, 22, 0, 120°; 4, 0, 45°.

3303. Base, diameter 12, center at c, 18, 0, 12. Vertex, point o, 2, 12, 22. Line A, 9, 20, 0°; 9, 0, 120°.

3304. Base, diameter 14, center at c, 22, 11, 0. Vertex, point o, 2, 12, 10. Line A, 2, 28, 150°; 2, 18, 135°.

3305. Base, diameter 12, center at c, 8, 12, 0. Vertex, point o, 22, 4, 15. Line ab; a, 19, − 7, 0; b, 19, − 2, − 13.

3306. Base, diameter 12, center at c, 9, 15, 0. Vertex, point o, 24, 3, 19. Line A, x, 25, 0°; x, 14, 0°.

CASE II. The base of the cone is in a plane perpendicular to H or V. Each (3).

3307. Base, center at c, 8, 10, 9; plan, a straight line, slope 45°; elevation, a circle, diameter 12. Vertex, point o, 23, 4, 9. Line A, 24, 0, 135°; 24, 0, 135°.

3308. Base, center at c, 8, 14, 16; plan, a straight line, slope 30°; elevation, a circle, diameter 12. Vertex, point o, 24, 2, 8. Line A, 24, 10, 150°; 8, 0, 45°.

3309. Base, center at c, 18, 10, 11; plan, a circle, diameter 12; elevation, a straight line, slope 150°. Vertex, point o, 2, 10, 3. Line A, 2, 24, 150°; 2, 16, 0°.

3310. Base, center at c, 24, 16, 6; plan, a circle, diameter 12; elevation, a straight line, slope 30°. Vertex, point o, 10, 4, 16. Line ab; a, 28, 3, 12; b, 28, -2, 4.

3311. Base, center at c, 11, 9, 14; plan, a straight line, slope 45°; elevation, a circle, diameter 14. Vertex, point o, 31, 6, 2. Line A, x, 20, 0°; x, 22, 0°.

3312. Base, a circle, diameter 12, lying in a plane perpendicular to H and V; center at c, 16, 10, 14. Vertex, point o, 2, 10, 14. Line A, 2, 24, 150°; 16, 0, 150°.

Problem 34. The given projection is that of two points on the surface of a cylinder. Find the other projections of these points, and pass a tangent plane at each point.

Case I. The base of the cylinder is a circle lying in either H or V, according to the location given for its center.

3401. (6) Base, diameter 10, center at c, 17, 0, 8. Elements, plan 150°, elevation 15°. Points a and b, 7, 8, z.

3402. (6) Base, diameter 10, center at c, 8, 7, 0. Elements, plan 0°, elevation 45°. Points a and b, 17, y, 6.

3403. (3) Base, diameter 14, center at c, 13, 14, 0. Elements, plan 30°, elevation 45°. Points a and b, 17, y, 10.

3404. (3) Base, diameter 12, center at c, 10, 0, 18. Elements, plan 45°, elevation 135°. Points a and b, 18, y, 8.

3405. (3) Base, diameter 14, center at c, 14, 0, 9. Elements, plan 45°, elevation 60°. Points a and b, 20, y, 28.

3406. (3) Base, diameter 14, center at c, 22, 0, 14. Elements, plan 120°, elevation 0°. Points a and b, 11, y, 10.

Case II. The base of the cylinder is a circle lying in P.

3407. (6) A cylinder of revolution, length 14. Left-hand base, diameter 10, center at c, 17, 9, 9. Points a and b, 26, y, 6.

3408. (6) A cylinder of revolution, length 15. Right-hand base, diameter 12, center at c, 17, 9, 7. Points a and b, 13, 6, z.

Case III. The base of the cylinder is in a plane perpendicular to H or V. Each (3).

3409. Base, center at c, 18, 9, 7; plan, a circle, diameter 12; elevation, a straight line, slope 150°. Elements to right of base; plan 45°, elevation 60°. Points a and b, 22, y, 18.

3410. Base, center at c, 14, 10, 7; plan, a circle, diameter 12; elevation, a straight line, slope 30°. Elements to left of base; plan 150°, elevation 135°. Points a and b, 6, y, 22.

3411. Base, center at c, 14, 6, 12; plan, a straight line, slope 150°; elevation, a circle, diameter 14. Elements to right of base; plan 60°, elevation 0°. Points a and b, 22, y, 7.

3412. Base, center at c, 26, 18, 8; plan, a circle, diameter 12; elevation, a straight line, slope 45°. Elements to left of base; plan 45°, elevation 0°. Points a and b, 16, 6, z.

3413. Base, center at c, 16, 6, 10; plan, a straight line, slope 150°; elevation, a circle, diameter 14. Elements to right of base; plan 60°, elevation 60°. Points a and b, 4, y, 20.

3414. Base, center at c, 9, 6, 20; plan, a straight line, slope 135°; elevation, a circle, diameter 12. Elements to right of base; plan 45°, elevation 120°. Points a and b, 19, 10, z.

Problem 35. Pass planes tangent to the given cylinder through the given point. Two tangent planes in each example.

Case I. The base of the cylinder is a circle lying in either H or V, according to the location given for its center.

3501. (6) Base, diameter 10, center at c, 7, 0, 6. Elements, plan 60°, elevation 30°. Point a, 24, − 9, 2.

3502. (6) Base, diameter 12, center at c, 9, 10, 0. Elements, plan 150°, elevation 45°. Point a, 26, 2, 6.

3503. (6) Base, diameter 12, center at c, 13, 0, 8. Elements, plan 60°, elevation 60°. Point a, 25, 3, 13.

3504. (3) Base, diameter 14, center at c, 11, 12, 0. Elements, plan 45°, elevation 60°. Point a, 27, 22, 8.

3505. (3) Base, diameter 14, center at c, 14, 0, 14. Elements, plan 45°, elevation 0°. Point a, 26, 8, 26.

3506. (3) Base, diameter 12, center at c, 15, 0, 16. Elements, plan 45°, elevation 120°. Point a, 7, − 7, − 4.

Case II. The base of the cylinder is in a plane perpendicular to H or V. Each (3).

3507. Base, center at c, 24, 4, 11; plan, a straight line, slope 30°; elevation, a circle, diameter 12. Elements to left of base, plan 150°, elevation 45°. Point a, 6, − 2, 16.

3508. Base, center at c, 9, 10, 6; plan, a circle, diameter 14; elevation, a straight line, slope 150°. Elements to right of base, plan 30°, elevation 60°. Point a, 29, 10, 11.

3509. Base, center at c, 11, 8, 14; plan, a circle, diameter 12;

elevation, a straight line, slope 135°. Elements to left of base, plan 135°, elevation 30°. Point *a*, 17, 14, 2.

3510. Base, center at *c*, 23, 12, 16; plan, a straight line, slope 30°; elevation, a circle, diameter 14. Elements to left of base, plan 150°, elevation 0°. Point *a*, 5, 10, 3.

3511. A cylinder of revolution, length 15, diameter 12, axis parallel to *GL*. Center of right-hand base at *c*, 17, 9, 12. Point *a*, 5, 11, 2.

3512. Base, a circle, diameter 12, lying in a plane perpendicular to *H* and *V*. Center at *c*, 15, 10, 12. Elements to left of base, plan 135°, elevation 120°. Point *a*, 7, 16, 17.

Problem 36. Pass planes tangent to the given cylinder parallel to the given line. Two tangent planes in each example.

Case I. The base of the cylinder is a circle lying in either *H* or *V*, according to the location given for its center. Each (3).

3601. Base, diameter 14, center at *c*, 24, 12, 0. Elements, plan 120°, elevation 135°. Line *A*, 2, 0, 60°; 5, 0, 30°.

3602. Base, diameter 14, center at *c*, 11, 11, 0. Elements, plan 45°, elevation 30°. Line *A*, 31, 6, 0°; 31, 0, 150°.

3603. Base, diameter 12, center at *c*, 12, 0, 13. Elements, plan 45°, elevation 0°. Line *A*, 31, 0, 150°; 22, 0, 45°.

3604. Base, diameter 14, center at *c*, 16, 13, 0. Elements, plan 0°, elevation 120°. Line *A*, 6, 0, 120°; 6, 8, 0°.

3605. Base, diameter 14, center at *c*, 23, 0, 16. Elements, plan 150°, elevation 60°. Line *A*, *x*, 16, 0°; *x*, 20, 0°.

3606. Base, diameter 12, center at *c*, 16, 0, 14. Elements, plan 45°, elevation 150°. Line *ab*; *a*, 5, 10, 61; *b*, 5, 3, 4.

Case II. The base of the cylinder is a circle lying in *P*. Each (3).

3607. Base, diameter 12, center at *c*, 17, 8, 12. Elements to left of base, plan 0°, elevation 0°. Line *A*, 13, 0, 30°; 5, 0, 60°.

3608. Base, diameter 12, center at *c*, 16, 10, 9. Elements to left of base, plan 0°, elevation 135°. Line *A*, 2, 0, 60°; 7, 0, 120°.

Case III. The base of the cylinder is in a plane perpendicular to *H* or *V*. Each (3).

3609. Base, center at *c*, 9, 7, 8; plan, a straight line, slope 150°; elevation, a circle, diameter 12. Elements to right of base, plan 30°, elevation 30°. Line *A*, 31, 0, 150°; 23, 0, 120°.

3610. Base, center at *c*, 12, 4, 11; plan, a straight line, slope 150°; elevation, a circle, diameter 12. Elements to right of base, plan 30°, elevation 45°. Line *A*, 15, 0, 60°; 31, 0, 135°.

3611. Base, center at c, 15, 11, 6; plan, a circle, diameter 12; elevation, a straight line, slope 135°. Elements to right of base, plan 120°, elevation 30°. Line A, 23, 0, 60°; 31, 0, 135°.

3612. Base, center at c, 9, 8, 8; plan, a circle, diameter 12; elevation, a straight line, slope 150°. Elements to right of base, plan 30°, elevation 45°. Line A, 31, 0, 120°; 31, 0, 150°.

3613. Base, center at c, 18, 8, 8; plan, a circle, diameter 12; elevation, a straight line, slope 45°. Elements to left of base, plan 135°, elevation 150°. Line ab; a, 14, 0, 10; b, 14, 8, 2.

Problem 37. The given projection is that of two points on the surface of the given sphere. Find the other projections of these points, and pass a tangent plane at each point. Each (3).

3701. Sphere, diameter 16, center at o, 12, 12, 14. Points a and b, 18, y, 11.

3702. Sphere, diameter 14, center at o, 14, 12, 15. Points a and b, 17, 7, z.

3703. Sphere, diameter 16, center at o, 19, 16, 13. Points a and b, 14, 13, z.

3704. Sphere, diameter 14, center at o, 22, 10, 9. Points a and b, 18, 14, z.

3705. Sphere, diameter 14, center at o, 23, 13, 15. Points a and b, 23, y, 11.

3706. Sphere, diameter 16, center at o, 9, 10, 8. Points a and b, 13, 6, z.

Problem 38. The given projection is that of a point or points (two, three, or four) on the surface of the given torus. Find the missing projections, and pass a tangent plane at each point. Each (2).

3801. Torus, axis vertical, center at o, 16, 16, 14. Outer diameter 30, inner diameter 10. Point a, 25, 6, z. Also, point c, 4, 7, z.

3802. Torus, axis vertical, center at o, 16, 16, 7. Outer diameter 28, inner diameter 8. Point a, 12, 10, z. Also, point c, 26, y, 11.

3803. Torus, axis vertical, center at o, 31, 14, 5. Outer diameter 28, inner diameter 12. Point a, 25, y, 8.

3804. Torus, axis perpendicular to V, center at o, 20, 12, 16. Outer diameter 28, inner diameter 8. Point a, 24, y, 9.

3805. Torus, axis perpendicular to V, center at o, 21, 13, 16. Outer diameter 26, inner diameter 10. Point a, 26, y, 20. Also, point c, 33, 13, z.

3806. Torus, axis perpendicular to V, center at o, 35, 10, 14. Outer diameter 24, inner diameter 8. Point a, 30, y, 23.

3807. Torus, axis parallel to GL, center at o, 30, 13, 14. Outer diameter 20, inner diameter 4. Point a, 27, y, 19.

Problem 39. Pass planes tangent to the sphere through the given line. Two results in each example. Each (2).

3901. Sphere, diameter 12, center at *o*, 34, 8, 10. Line *A*, 29, 0, 135°; 24, 0, 120°.

3902. Sphere, diameter 12, center at *o*, 19, 10, 14. Line *A*, 38, 0, 120°; 24, 0, 45°.

3903. Sphere, diameter 16, center at *o*, 19, 12, 9. Line *A*, 26, 0, 45°; 30, 0, 60°.

3904. Sphere, diameter 10, center at *o*, 14, 9, 7. Line *A*, 20, 0, 60°; 6, 0, 30°.

3905. Sphere, diameter 14, center at *o*, 10, 8, 10. Line *ab*; *a*, 26, 21, 11; *b*, 26, – 7, – 5.

3906. Sphere, diameter 16, center at *o*, 40, 12, 12. Line *ab*; *a*, 24, 3, 11; *b*, 24, 15, 4.

SPECIAL CASE. Each (2).

3907. Sphere, diameter 14, center at *o*, 18, 12, 19. Line *A*, 10, 0, 135°; 10, – 17, 0°.

3908. Sphere, diameter 12, center at *o*, 25, 16, 10. Line *A*, 29, 0, 150°; 29, 7, 0°.

Problem 40. Pass planes tangent to the given double curved surface of revolution through the given line. Two results in each example.

CASE I. Each (2).

4001. A sphere, diameter 16, center at *o*, 25, 16, 14. Line *A*, 25, 16, 60°; 25, 2, 30°.

4002. A solid torus, axis vertical, center at *o*, 21, 14, 8. Outer diameter 24, thickness 8. Line *A*, 35, 0, 135°; 45, 0, 135°.

4003. An ellipsoid, axis vertical, center at *o*, 25, 14, 12. Principal meridian section, an ellipse, axes 18 and 12, major axis parallel to *GL*. Line *A*, 11, 0, 45°; 40, 0, 165°.

4004. An ellipsoid, axis perpendicular to *V*, center at *o*, 20, 8, 16. Principal meridian section, an ellipse, axes 20 and 10, major axis parallel to *GL*. Line *A*, 20, 24, 135°; 20, 16, 120°.

CASE II. Each (2).

4005. A solid torus, axis vertical, center at *o*, 25, 20, 18. Outer diameter 24, thickness 12. Line *A*, 31, 4, ⊙; 31, 0, 90°.

4006. An ellipsoid, axis perpendicular to *V*, center at *o*, 17, 16, 16. Principal meridian section, an ellipse, axes 24 and 12, major axis parallel to *GL*. Line *A*, 35, 0, 90°; 35, 12, ⊙.

4007. An ellipsoid, axis parallel to *H* and *V* (perpendicular to *P*), center at *o*, 25, 10, 16. Meridian section, an ellipse, axes 24 and 14, major axis parallel to *GL*. Line *A*, *x*, 13, 0°; *x*, 5, 0°.

CASE III. Each (2).

4008. An ellipsoid, axis vertical, center at *o*, 28, 16, 16. Principal meridian section, an ellipse, axes 24 and 14, minor axis parallel to *GL*. Line *A*, 28, 0, 45°; 28, 10, 0°.

4009. An ellipsoid, axis vertical, center at *o*, 27, 16, 10. Principal meridian section, an ellipse, axes 20 and 12, major axis parallel to *GL*. Line *A*, 23, 0, 60°; 23, 27, 0°.

4010. An ellipsoid, axis perpendicular to *V*, center at *o*, 29, 12, 12. Principal meridian section, an ellipse, axes 20 and 14, minor axis parallel to *GL*. Line *A*, 13, 30, 0°; 13, 0, 45°.

4011. A solid torus, axis perpendicular to *V*, center at *o*, 32, 9, 16. Outer diameter 22, thickness 8. Line *A*, 32, 32, 0°; 32, 12, 30°.

4012. A solid torus, axis vertical, center at *o*, 18, 16, 10. Outer diameter 28, thickness 8. Line *A*, *x*, 21, 0°; *x*, 2, 0°.

Problem 41. Find the intersection of the cone and plane. Choose elements at intervals of 4 to 8 units on the base, and at least 8 elements in each example. The chosen elements should include the two contour elements of each projection. Draw a line tangent to the intersection at some point not on a contour element. Develop the entire curved surface of the cone. On this development locate the section and the tangent line. Place the first element of the development on a vertical line, 2 units inside the right-hand edge of the allotted space. Each (1).

CASE I. The base of the cone is a circle lying in *H* or *V*, according to the location of its center.

4101. Cone, diameter of base 20, center at *c*, 32, 12, 0. Vertex, point *o*, 2, 29, 26. Plane *Q*; *HQ*, 2, 12, 30°; *VQ*, 2, 12, 30°.

4102. Cone, diameter of base 18, center at *c*, 35, 0, 18. Vertex, point *o*, 7, 22, 23. Plane *Q*, 0, 30°, 45°.

4103. Cone, diameter of base 24, center at *c*, 14, 14, 0. Vertex, point *o*, 45, 24, 23. Plane *Q*, 23, 60°, 135°.

4104. Cone, diameter of base 24, center at *c*, 34, 16, 0. Vertex, point *o*, 2, 16, 24. Plane *Q*, 29, 135°, 45°.

4105. Cone, diameter of base 22, center at *c*, 14, 14, 0. Vertex, point *o*, 35, 14, 18. Plane *Q*, 49, 135°, 150°.

4106. Cone, diameter of base 24, center at *c*, 34, 17, 0. Vertex, point *o*, 2, 7, 20. Plane *Q*, ∞, 32, 12.

CASE II. The base of the cone is a circle lying in *P*.

4107. Cone, diameter of base 24, center at *c*, 28, 14, 14. Vertex, point *o*, 56, 14, 14. Plane *Q*, 65, 135°, 120°.

4108. Same cone as in the preceding example. Plane *Q*, 38, 60°, 135°.

Case III. The base of the cone lies in a plane perpendicular to H or V.

4109. Base of cone, center at c, 31, 3, 12; plan, a straight line, slope 15°; elevation, a circle, diameter 20. Vertex, point o, 3, 24, 16. Plane Q, 2, 45°, 45°.

4110. Base of cone, center at c, 14, 14, 22; plan, a circle, diameter 20; elevation, a straight line, slope 30°. Vertex, point o, 37, 26, 0. Plane Q, 6, 135°, 30°.

4111. Base of cone, center at c, 14, 15, 6; plan, a circle, diameter 20; elevation, a straight line, slope 150°. Vertex, point o, 46, 5, 24. Plane Q, 43, 30°, 150°.

4112. Base of cone, center at c, 32, 9, 14; plan, a straight line, slope 30°; elevation, a circle, diameter 24. Vertex, point o, 2, 30, 20. Plane Q, ∞, 22, 32.

Problem 42. Find the intersection of the conical frustum and the given plane. Choose elements at intervals of 6 to 10 units on the larger base, and at least 8 elements in each example. The chosen elements should include the two contour elements of each projection. Draw a line tangent to the intersection at some point not on a contour element. Develop the curved surface of the frustum. On this development locate the section and the tangent line. Place the first element of the development on a vertical line whose distance, x, from the left-hand edge of the allotted space is given.

In all but the last example the base first located is a circle lying in H or V, according to the location of its center. The other base is parallel to the first. Each (1).

4201. Large base, diameter 18, center at c, 40, 23, 0. Small base, diameter 10, center at e, 11, 6, 22. Plane Q, 48, 135°, 135°. $x = 55$.

4202. Large base, diameter 20, center at c, 38, 14, 0. Small base, diameter 14, center at e, 10, 24, 20. Plane Q, 25, 120°, 30°. $x = 51$.

4203. Large base, diameter 22, center at c, 36, 12, 0. Small base, diameter 12, center at e, 8, 24, 22. Plane Q, 42, 45°, 165°. $x = 98$.

4204. Large base, diameter 20, center at c, 38, 0, 20. Small base, diameter 14, center at e, 10, 18, 12. Plane Q, 38, 30°, 135°. $x = 51$.

4205. Large base, diameter 20, center at c, 12, 0, 18. Small base, diameter 14, center at e, 42, 20, 18. Plane Q, 40, 135°, 120°. $x = 51$.

4206. Small base, diameter 12, center at e, 41, 0, 8. Large base, diameter 20, center at c, 14, 24, 20. Plane Q, 0, 30°, 45°. $x = 98$.

4207. Each base lies in a plane perpendicular to H and V. Left-hand base, diameter 20, center at c, 30, 14, 18. Right-hand base, diameter 14, center at e, 56, 14, 18. Place the central element of the development horizontal, and in the middle of the sheet.

Problem 43. Find the intersection of the cylinder and plane Q. Take elements equally spaced on that projection of the base which is a circle, and select them so as to include the two contour elements of each projection. Draw a line tangent to the intersection at some point not on a contour element. Develop the curved surface of the cylinder between the base and the section Q, and locate the tangent line on this development. The plane R, given in Cases I and III, should be used in obtaining the right section. Place the first element of the development on a vertical line whose distance, x, from the left-hand edge of the allotted space is given. Each (1).

CASE I. The base of the cylinder is a circle lying in H or V, according to the location of its center.

4301. Base, diameter 14, center at c, 15, 9, 0. Elements, plan 30°, elevation 60°. Plane Q, 32, 75°, 120°. Plane R, 48, 120°, 150°. $x = 55$.

4302. Base, diameter 16, center at c, 12, 0, 24. Elements, plan 45°, elevation 150°. Plane Q, 38, 135°, 120°. Plane R, 27, 135°, 60°. $x = 56$.

4303. Base, diameter 14, center at c, 14, 0, 10. Elements, plan 45°, elevation 45°. Plane Q, 56, 135°, 120°. Plane R, 41, 135°, 135°. $x = 60$.

4304. Base, diameter 16, center at c, 32, 0, 10. Elements, plan 120°, elevation 135°. Plane Q, 20, 45°, 120°. Plane R, 6, 30°, 45°. $x = 50$.

4305. Base, diameter 16, center at c, 40, 20, 0. Elements, plan 0°, elevation 150°. Plane Q, 13, 60°, 120°. Plane R, 26, 90°, 60°. $x = 55$.

4306. Base, diameter 12, center at c, 12, 8, 0. Elements, plan 45°, elevation 60°. Plane Q, 56, 150°, 120°. Plane R, 36, 135°, 150°. $x = 62$.

CASE II. The cylinder is one of revolution, with its axis parallel to GL.

4307. Left-hand base, diameter 14, center at c, 21, 9, 14. Length of cylinder 24. Plane Q, 15, 60°, 45°. $x = 51$.

4308. Left-hand base, diameter 16, center at c, 23, 10, 20. Length of cylinder 24. Plane Q, 10, 30°, 60°. $x = 49$.

4309. Left-hand base, diameter 14, center at c, 27, 14, 18. Length of cylinder 26. Plane Q, 14, 120°, 30°. $x = 55$.

CASE III. The base of the cylinder lies in a plane perpendicular to H or V.

4310. Base, center at c, 18, 4, 22; plan, a straight line, slope 165°; elevation, a circle, diameter 14. Elements to right of base, plan 60°, elevation 135°. Plane Q, 36, 120°, 45°. Plane R, 27, 150°, 45°. $x = 50$.

4311. Base, center at c, 10, 10, 6; plan, a circle, diameter 16; elevation, a straight line, slope 150°. Elements to right of base, plan 30°, elevation 45°. Plane Q, 52, 150°, 120°. Plane R, 52, 120°, 135°. $x = 49$.

4312. Base, center at c, 30, 5, 8; plan, a straight line, slope 30°; elevation, a circle, diameter 14. Elements to left of base, plan 120°, elevation 120°. Plane Q, ∞, 30, 28. Plane R; HR, 30, 28, 30°; VR, 30, 28, 30°. $x = 52$.

Problem 44. Find the intersection of the given pyramid and plane. In each example point o is the vertex of the pyramid, and abc or $abcd$ the base. Each (3).

4401. Pyramid $oabc$; o, 28, 24, 24; a, 2, 0, 8; b, 8, 0, 16; c, 20, 0, 4. Plane Q, 32, 150°, 135°.

4402. Pyramid $oabcd$; o, 28, 28, 28; a, 1, 5, 0; b, 12, 5, 0; c, 20, 9, 0; d, 7, 16, 0. Plane Q, 33, 120°, 150°.

4403. Pyramid $oabcd$; o, 2, 25, 5; a, 11, 0, 23; b, 20, 0, 28; c, 32, 0, 24; d, 25, 0, 16. Plane Q, 8, 150°, 30°.

4404. Pyramid $oabc$; o, 30, 6, 3; a, 4, 26, 18; b, 4, 17, 9; c, 4, 6, 26. Plane Q, 27, 105°, 120°.

4405. Pyramid $oabc$; o, 5, 26, 24; a, 14, 3, 0; b, 26, 5, 3; c, 30, 17, 4. Plane Q, 1, 45°, 30°.

4406. Pyramid $oabcd$; o, 3, 3, 6; a, 15, 30, 27; b, 25, 24, 27; c, 30, 21, 19; d, 20, 27, 17. Plane Q, ∞, 22, 30.

Find the intersection of the given prism and plane. In each example but one base of the prism is located, and the prism is indefinite in length. Each (3).

4407. Base abc; a, 12, 0, 8; b, 21, 0, 2; c, 32, 0, 6. Lateral edges to left of base, plan 120°, elevation 120°. Plane Q, 0, 60°, 45°.

4408. Base $abcd$; a, 14, 0, 26; b, 25, 0, 28; c, 32, 0, 18; d, 20, 0, 18. Lateral edges to left of base, plan 135°, elevation 30°. Plane Q, 29, 30°, 135°.

4409. Base $abcd$; a, 2, 5, 0; b, 15, 3, 0; c, 19, 12, 0; d, 6, 16, 0. Lateral edges to right of base, plan 45°, elevation 60°. Plane Q, ∞, 28, 22.

4410. Base $abcd$; a, 26, 11, 23; b, 26, 21, 14; c, 26, 3, 11; d, 26, 13, 4. Lateral edges to left of base, plan 150°, elevation 15°. Plane Q, 4, 45°, 150°.

4411. Base abc; a, 2, 9, 12; b, 7, 6, 0; c, 17, 0, 4. Lateral edges to right of base, plan 60°, elevation 45°. Plane Q, 32, 135°, 120°.

4412. Base abc; a, 2, 13, 7; b, 11, 13, 4; c, 17, 4, 2. Lateral edges to right of base, plan 45°, elevation 30°. Plane Q, 31, 135°, 135°.

Problem 45. Find the intersection of the double curved surface of revolution and the given plane.

First Construction. Use as auxiliaries the principal meridian plane, the meridian plane of symmetry, and a sufficient number of planes perpendicular to the axis of the surface. Each (2).

4501. An ellipsoid. Axis vertical. Center at o, 29, 15, 16. Principal meridian section, an ellipse, axes 26 and 16, major axis parallel to GL. Plane Q, 12, 120°, 30°.

4502. A circular spindle. Axis perpendicular to V. Center at o, 30, 16, 15. Greatest diameter, 16. Meridian section, two arcs of circles, radius 12. Plane Q, 8, 60°, 45°.

4503. An ellipsoid. Axis vertical. Center at o, 25, 16, 16. Principal meridian section, an ellipse, axes 28 and 20, minor axis parallel to GL. Plane Q, 26, 135°, 60°.

4504. An ellipsoid. Axis vertical. Center at o, 25, 12, 15. Principal meridian section, an ellipse, axes 26 and 18, minor axis parallel to GL. Plane Q, ∞, 24, 27.

Second Construction. Take a secondary ground line crossing the original GL at the point whose distance x is given. Use as auxiliaries the principal meridian plane, the meridian plane of symmetry, and a sufficient number of planes perpendicular to the axis of the surface. Draw a line tangent to the intersection at a point where neither projection of the tangent line can be drawn by inspection. This excludes points on the contour of either projection, and on the meridian plane of symmetry. Each (2).

4505. A torus. Axis vertical. Center at o, 14, 17, 7. Outer diameter 26, inner diameter 6. Plane Q, 50, 135°, 150°. $x = 34$.

4506. A torus. Axis vertical. Center at o, 22, 18, 10. Outer diameter 24, inner diameter 8. Plane Q, 24, 120°, 45°. $x = 4$.

4507. A torus. Axis vertical. Center at o, 34, 17, 11. Outer diameter 24, inner diameter 4. Plane Q, 2, 45°, 45°. $x = 42$.

4508. A torus. Axis vertical. Center at o, 18, 20, 12. Outer diameter 24, inner diameter 8. Plane Q, 20, 150°, 15°. $x = 16$.

4509. A torus. Axis vertical. Center at o, 25, 20, 6. Outer diameter 24, inner diameter 8. Plane Q, 27, 45°, 165°. $x = 31$.

Problems 46, 47, 48. Find the intersection of the two given curved surfaces. Draw a line tangent to the intersection at some point where neither projection will coincide with an element of one of the surfaces. This excludes all points on limiting planes, and any point which lies on any contour element of either surface.

The auxiliary planes chosen should contain all the contour elements of both surfaces which occur between the limiting planes. However,

do not take the intermediate planes too near together. An element which is practically, although not exactly, the true contour will serve, if by so doing two intermediate planes which are close together can be replaced by one midway between. This remark does not apply to the limiting planes, which should be accurately placed.

The base of each surface is a circle lying in *H* or *V*, according to the location of its center. Each (2).

4601. Cylinder *A*. Base, diameter 14, center at *c*, 17, 8, 0. Elements, plan 60°, elevation 60°. Cylinder *B*. Base, diameter 12, center at *e*, 35, 11, 0. Elements, plan 135°, elevation 120°.

4602. Cylinder *A*. Base, diameter 16, center at *c*, 16, 24, 0. Elements, plan 120°, elevation 45°. Cylinder *B*. Base, diameter 12, center at *e*, 35, 11, 0. Elements, plan 15°, elevation 135°.

4603. Cylinder *A*. Base, diameter 14, center at *c*, 12, 9, 0. Elements, plan 45°, elevation 45°. Cylinder *B*. Base, diameter 16, center at *e*, 37, 11, 0. Elements, plan 135°, elevation 150°.

4604. Cylinder *A*. Base, diameter 10, center at *c*, 8, 7, 0. Elements, plan 0°, elevation 45°. Cylinder *B*. Base, diameter 16, center at *e*, 38, 0, 10. Elements, plan 135°, elevation 135°.

4701. Cone *A*. Base, diameter 12, center at *c*, 24, 7, 0. Vertex, point *a*, 2, 26, 26. Cylinder *B*. Base, diameter 10, center at *e*, 35, 15, 0. Elements, plan 0°, elevation 150°.

4702. Cone *A*. Base, diameter 18, center at *c*, 11, 13, 0. Vertex, point *a*, 29, 26, 19. Cylinder *B*. Base, diameter 14, center at *e*, 30, 10, 0. Elements, plan 135°, elevation 135°.

4703. Cone *A*. Base, diameter 18, center at *c*, 10, 10, 0. Vertex, point *a*, 31, 17, 31. Cylinder *B*. Base, diameter 14, center at *e*, 29, 0, 9. Elements, plan 120°, elevation 120°.

4704. Cone *A*. Base, diameter 16, center at *c*, 9, 12, 0. Vertex, point *a*, 49, 4, 23. Cylinder *B*. Base, diameter 14, center at *e*, 23, 22, 0. Elements, plan 120°, elevation 60°.

4801. Cone *A*. Base, diameter 18, center at *c*, 10, 17, 0. Vertex, point *a*, 30, 22, 17. Cone *B*. Base, diameter 16, center at *e*, 29, 10, 0. Vertex, point *b*, 18, 28, 28.

4802. Cone *A*. Base, diameter 16, center at *c*, 14, 0, 20. Vertex, point *a*, 32, 30, 3. Cone *B*. Base, diameter 20, center at *e*, 34, 0, 21. Vertex, point *b*, 6, 30, 3.

4803. Cone *A*. Base, diameter 16, center at *c*, 21, 0, 9. Vertex, point *a*, 41, 32, 27. Cone *B*. Base, diameter 18, center at *e*, 39, 12, 0. Vertex, point *b*, 10, 7, 27.

4804. Cone *A*. Base, diameter 16, center at *c*, 14, 16, 0. Vertex, point *a*, 45, 15, 28. Cone *B*. Base, diameter 18, center at *e*, 32, 21, 0. Vertex, point *b*, 19, 4, 28.

Problem 49. Find the intersection of the given surfaces of revolution *A* and *B*. Each (3).

4901. *A*, a cone. Vertex, point *c*, 2, 16, 30. Base, diameter 20, center at *e*, 24, 16, 8. *B*, a circular spindle. Axis vertical. Center at *o*, 14, 16, 14. Greatest diameter, 14. Meridian section, two arcs of circles, radius 14.

4902. *A*, a cylinder. Diameter 14. Center of one base, point *c*, 9, 20, 26; of the other base, point *e*, 24, 20, 0. *B*, a circular spindle. Axis vertical. Center at *o*, 19, 20, 14. Greatest diameter, 14. Meridian section, two arcs of circles, radius 14.

4903. *A*, a cone. Vertex, point *c*, 5, 14, 2. Base, diameter 22, center at *e*, 25, 14, 22. *B*, an ellipsoid. Axis vertical. Center at *o*, 11, 14, 18. Principal meridian section, an ellipse, axes 24 and 20, minor axis parallel to *GL*.

4904. *A*, a cylinder. Diameter 20. Center of one base, point *c*, 6, 16, 22; of the other base, point *e*, 27, 16, 10. *B*, an ellipsoid. Axis vertical. Center at *o*, 15, 16, 15. Principal meridian section, an ellipse, axes 22 and 18, major axis parallel to *GL*.

4905. *A*, a cylinder. Diameter 12. Center of one base, point *c*, 13, 14, 6; of the other base, point *e*, 27, 14, 20. *B*, a torus. Axis vertical. Center at *o*, 15, 14, 16. Outer diameter 24, inner diameter 8.

COROLLARY. Find the intersection of the given surfaces of revolution *A* and *B*. Each (3).

4906. *A*, a cone. Vertex, point *a*, 14, 10, 22. Base, diameter 16, center at *b*, 14, 10, 0. *B*, a cone. Vertex, point *c*, 18, 13, 26. Base, diameter 18, center at *d*, 18, 13, 0.

4907. *A*, a cone. Vertex, point *c*, 19, 16, 32. Base, diameter 24, center at *e*, 19, 16, 0. *B*, an ellipsoid. Axis vertical. Center at *o*, 12, 12, 17. Principal meridian section, an ellipse, axes 24 and 20, minor axis parallel to *GL*.

4908. *A*, an ellipsoid. Axis vertical. Center at *e*, 14, 16, 16. Principal meridian section, an ellipse, axes 24 and 20, major axis parallel to *GL*. *B*, a sphere. Diameter 24; center at *o*, 20, 19, 13.

Problem 50. Find the intersection of the given sphere and cone. Each (2).

5001. Sphere. Diameter 24, center at *o*, 24, 13, 18. Cone. Base, a circle in *H*, diameter 18, center at *e*, 33, 10, 0. Vertex, point *c*, 2, 32, 23.

5002. Sphere. Diameter 16, center at *o*, 18, 19, 11. Cone. Base, a circle in *H*, diameter 18, center at *e*, 29, 23, 0. Vertex, point *c*, 4, 1, 24.

5003. Sphere. Diameter 22, center at *o*, 30, 14, 20. Cone. Base, center at *e*, 21, 22, 21; plan, a straight line, slope 30°; elevation, a circle, diameter 22. Vertex, point *c*, 48, 2, 0.

5004. Sphere. Diameter 26, center at o, 28, 16, 13. Cone. Base, a circle in H, diameter 20, center at e, 22, 11, 0. Vertex, point c, 46, 32, 30.

Problem 51. Find the intersection of the given sphere and cylinder. Each (2).

5101. Sphere. Diameter 22, center at o, 14, 22, 16. Cylinder. Base, a circle in H, diameter 18, center at e, 10, 10, 0. Elements, plan 45°, elevation 60°.

5102. Sphere. Diameter 24, center at o, 37, 14, 15. Cylinder. Base, center at e, 33, 23, 3; plan, a circle, diameter 18; elevation, a straight line, slope 165°. Elements to right of base, plan 120°, elevation 45°.

5103. Sphere. Diameter 22, center at o, 35, 21, 16. Cylinder. Base, center at e, 41, 10, 6; plan, a circle, diameter 14; elevation, a straight line, slope 30°. Elements to left of base, plan 120°, elevation 120°.

5104. Sphere. Diameter 22, center at o, 38, 14, 12. Cylinder. Base, center at e, 30, 22, 23; plan, a circle, diameter 14; elevation, a straight line, slope 30°. Elements to right of base, plan 135°, elevation 135°.

Problem 52. Find the intersection of the given surfaces A and B. Each (3).

5201. A, an oblique cylinder. Base, a circle in V; diameter 14, center at e, 9, 0, 8. Elements, plan 45°, elevation 60°. B, a torus. Axis vertical. Center at o, 19, 16, 20. Outer diameter, 24; inner diameter, 8.

5202. A, an oblique cylinder. Base, center at a, 7, 6, 4; plan, a circle, diameter 12; elevation, a straight line, slope 150°. B, the frustum of a cone of revolution. Smaller base, a circle in H, diameter 16, center at b, 18, 18, 0. Larger base, a circle parallel to H, diameter 26, center at c, 18, 18, 27.

5203. A, a surface of revolution. Axis vertical. Length of axis, 20; center at c, 16, 18, 14. Central (and least) diameter of surface, 8. Meridian section, two arcs of circles, radius 12, convex towards the axis. B, an ellipsoid of revolution. Axis perpendicular to V. Center at o, 22, 18, 13. Principal meridian section, an ellipse, axes 24 and 16, minor axis parallel to GL.

5204. A, an ellipsoid of revolution. Axis vertical. Center at o, 15, 18, 13. Principal meridian section, an ellipse, axes 24 and 20, major axis parallel to GL. B, a circular spindle. Axis perpendicular to V. Center at c, 22, 14, 18. Greatest diameter, 18. Meridian section, two arcs of circles, radius 18.

5205. *A*, an ellipsoid of revolution. Axis vertical. Center at *o*, 15, 16, 32. Principal meridian section, an ellipse, axes 24 and 14, major axis parallel to *GL*. *B*, a cone of revolution. Base, a circle in *V*, diameter 20, center at *e*, 20, 0, 12. Vertex, point *c*, 20, 32, 12.

5206. *A*, a torus. Axis vertical. Center at *o*, 16, 17, 15. Outer diameter, 28; inner diameter, 12. *B*, an ellipsoid of revolution. Axis perpendicular to *V*. Center at *c*, 19, 7, 15. Principal meridian section, an ellipse, axes 24 and 16, major axis parallel to *GL*.

Problem 53. Given the linear directrices and plane directrix of a hyperbolic paraboloid, and one projection of a point in the surface. Find the other projection of the point. Pass a plane tangent to the surface at this point. Each (2).

5301. Linear directrices, *A*, 8, 0, 45°; 10, 0, 60°, and *B*, 30, 0, 30°; 33, 0, 60°. Plane directrix, *H*. Point *c*, 24, *y*, 15.

5302. Linear directrices, *A*, 3, 4, 0°; 3, 0, 45°, and *B*, 47, 0, 150°; 25, 0, 45°. Plane directrix, *H*. Point *c*, 19, *y*, 3.

5303. Linear directrices, *A*, 2, 0, 30°; 2, 0, 60°, and *B*, 26, 0, 30°; 46, 0, 135°. Plane directrix, *V*. Point *c*, 22, 8, *z*.

5304. Linear directrices, *A*, 5, 0, 60°; 3, 0, 45°, and *B*, 32, 0, 45°; 31, 0, 60°. Plane directrix, *V*. Point *c*, 25, *y*, 11.

5305. Linear directrices, *A*, 9, 8, 30°; 9, 0, 60°, and *B*, 33, 0, 30°; 47, 0, 135°. Plane directrix, *X*, 16, 150°, 90°. Point *c*, 25, *y*, 19.

5306. Linear directrices, *A*, 4, 0, 15°; 37, 0, 135°, and *B*, 11, 0, 75°; 24, 0, 135°. Plane directrix, *X*, 16, 135°, 90°. Point *c*, 19, 8, *z*. Also, point *d*, 23, *y*, 5.

Problem 54. Given the linear directrices and plane directrix of a hyperbolic paraboloid; also a line which intersects the surface. Find the tangent plane which contains the given line and an element parallel to the given plane directrix. Find also the point of tangency of this plane. Each (2).

5401. Linear directrices, *A*, 3, 0, 45°; 5, 0, 60°, and *B*, 39, 0, 120°; 27, 0, 45°. Plane directrix, *H*. Line *C*, 21, 0, 60°; 34, 0, 135°.

5402. Linear directrices, *A*, 25, 28, 30°; 25, 0, 150°, and *B*, 25, 4, 30°; 25, 0, 30°. Plane directrix, *X*, 19, 150°, 90°. Line *C*, 17, 0, 60°; 29, 0, 135°.

5403. Linear directrices, *A*, 29, 0, 150°; 23, 0, 120°, and *B*, 41, 0, 150°; 45, 0, 120°. Plane directrix, *H*. Line *C*, 10, 0, 45°; 16, 0, 60°.

5404. Linear directrices, *A*, 2, 0, 60°; 22, 0, 135°, and *B*, 34, 0, 120°; 36, 0, 135°. Plane directrix, *V*. Line *C*, 8, 0, 45°; 39, 0, 150°.

5405. Linear directrices, *A*, 28, 0, 165°; 7, 0, 60°, and *B*, 45, 0, 120°; 27, 0, 60°. Plane directrix, *H*. Line *C*, 17, 13, 0°; 17, 0, 45°.

5406. Linear directrices, *A*, 23, 0, 120°; 9, 0, 45°, and *B*, 25, 0, 45°; 27, 0, 45°. Plane directrix, *H*. Line *C*, 19, 0, 60°; 32, 0, 135°.

Problem 55. Given a hyperbolic paraboloid, defined by two linear directrices and a plane directrix. Find the intersection of the surface with the given plane. Draw a tangent line at some point of the intersection. Each (2).

5501. Linear directrices, *A*, 5, 0, 45°; 5, 0, 30°, and *B*, 24, 0, 30°; 29, 0, 45°. Plane directrix, *H*. Plane *Q*, 27, 60°, 135°.

5502. Linear directrices, *A*, 27, 0, 135°; 26, 0, 120°, and *B*, 30, 0, 60°; 46, 0, 120°. Plane directrix, *V*. Plane *Q*, 14, 30°, 120°.

5503. Linear directrices, *A*, 5, 0, 45°; 9, 0, 60°, and *B*, 24, 0, 60°; 39, 0, 135°. Plane directrix, *H*. Plane *Q*, 29, 60°, 135°.

5504. Linear directrices, *A*, 24, 0, 120°; 20, 0, 135°, and *B*, 44, 4, 0°; 44, 0, 135°. Plane directrix, *H*. Plane *Q*, 9, 45°, 60°.

5505. Linear directrices, *A*, 4, 0, 60°; 6, 0, 60°, and *B*, 47, 0, 135°; 34, 0, 60°. Plane directrix, *H*. Plane *Q*, 3, 60°, 45°.

5506. Linear directrices, *A*, 4, 0, 60°; 2, 0, 45°, and *B*, 28, 0, 45°; 34, 0, 60°. Plane directrix, *V*. Plane *Q*, 22, 135°, 30°.

Problem 56. A hyperboloid of revolution is defined by its axis and generating line. The given projection is that of two points on the surface. Find the other projections of these points, and pass a tangent plane at each point. Each (2).

5601. Axis vertical. Center at *o*, 28, 16, *z*. Generating line, *B*, 42, 13, 0°; 42, 0, 135°. Points *c* and *d*, 35, *y*, 24.

5602. Axis vertical. Center at *o*, 18, 18, *z*. Generating line, *B*, 10, 22, 0°; 10, 0, 60°. Points *c* and *d*, 13, 13, *z*.

5603. Axis vertical. Center at *o*, 28, 16, *z*. Generating line, *B*, 41, 12, 0°; 41, 0, 135°. Points *c* and *d*, 24, *y*, 8.

5604. Axis perpendicular to *V*. Center at *o*, 30, *y*, 14. Generating line, *B*, 22, 0, 60°; 22, 9, 0°. Points *c* and *d*, 34, *y*, 8.

5605. Axis perpendicular to *V*. Center at *o*, 20, *y*, 16. Generating line, *B*, 10, 0, 60°; 24, 0, 120°. Points *c* and *d*, 26, *y*, 24.

5606. Axis perpendicular to *V*. Center at *o*, 27, *y*, 15. Generating line, *B*, 21, 0, 60°; 10, 0, 30°. Points *c* and *d*, 21, 23, *z*.

Problem 57. A hyperboloid of revolution is defined by its axis and generating line. The given line intersects the surface. Pass tangent planes through this line, and find their points of tangency. Two results in each example. Each (2).

5701. Axis vertical. Center at *o*, 25, 17, *z*. Generating line, *B*, 41, 0, 150°; 13, 0, 45°. Line *A*, 9, 0, 30°; 12, 0, 60°.

5702. Axis vertical. Center at *o*, 15, 16, *z*. Generating line, *B*, 6, 10, 0°; 6, 0, 60°. Line *A*, 28, 0, 135°; 28, 28, 0°.

5703. Axis perpendicular to V. Center at o, 30, y, 16. Generating line, B, 41, 0, 120°; 10, 0, 45°. Line A, 12, 0, 45°; 28, 0, 45°.

5704. Axis perpendicular to V. Center at o, 17, y, 16. Generating line, B, 7, 0, 45°; 17, 11, 30°. Line A, 31, 0, 120°; 5, 0, 30°.

Problem 58. A hyperboloid of revolution is defined by its axis and generating line. Find the intersection of the given plane with the surface. Draw a line tangent to the intersection at some point not in the meridian plane of symmetry. Each (2).

5801. Axis vertical. Center at o, 19, 16, z. Generating line, B, 6, 20, 0°; 6, 0, 45°. Plane Q, 47, 120°, 150°.

5802. Axis vertical. Center at o, 27, 16, z. Generating line, B, 41, 12, 0°; 41, 0, 135°. Plane Q, 18, 120°, 30°.

5803. Axis vertical. Center at o, 21, 14, z. Generating line, B, 9, 21, 0°; 9, 0, 45°. Plane Q, 25, 30°, 135°.

5804. Axis vertical. Center at o, 29, 16, z. Generating line, B, 17, 16, 30°; 39, 0, 135°. Plane Q, 21, 135°, 45°.

5805. Axis perpendicular to V. Center at o, 21, y, 16. Generating line, B, 8, 0, 45°; 8, 21, 0°. Plane Q, 47, 135°, 120°.

5806. Axis perpendicular to V. Center at o, 24, y, 14. Generating line, B, 27, 0, 120°; 0, 0, 45°. Plane Q, 34, 150°, 45°.

Problem 59. Pass planes tangent to the double curved surface of revolution through the given line. Two results in each example. Each (2).

5901. An ellipsoid. Axis vertical. Center at o, 20, 15, 16. Principal meridian section, an ellipse, axes 26 and 16, minor axis parallel to GL. Line A, 21, 0, 45°; 44, 0, 150°.

5902. An ellipsoid. Axis vertical. Center at o, 30, 17, 14. Principal meridian section, an ellipse, axes 24 and 12, major axis parallel to GL. Line A, 23, 0, 120°; 29, 0, 135°.

5903. A solid torus. Axis vertical. Center at o, 15, 17, 12. Outer diameter 28, thickness 12. Line A, 27, 0, 60°; 45, 0, 135°.

5904. An ellipsoid. Axis perpendicular to V. Center at o, 32, 10, 13. Principal meridian section, an ellipse, axes 22 and 14, major axis parallel to GL. Line A, 40, 0, 150°; 22, 0, 135°.

5905. A circular spindle. Axis vertical. Center at o, 30, 12, 12. Greatest diameter, 12. Meridian section, two arcs of circles, radius 10. Line A, 20, 21, 0°; 20, 0, 45°.

5906. A solid torus. Axis vertical. Center at o, 13, 18, 9. Outer diameter 22, thickness 10. Line ab; a, 27, 21, 7; b, 27, 0, 0.

Problem 60. A helix is defined by its axis, pitch, and generating point. Draw one complete turn of the helix, dividing the pitch into 12 equal parts. Draw a tangent line at each of the given points on the curve. Each (2).

6001. Helix is left handed. Axis, line A, 25, 17, ⊙; 25, 0, 90°. Pitch, 24. Generating point, o, 41, 17, 0. Point c, x, y, 4. Point d, x, y, 10.

6002. Helix is left handed. Axis, line A, 25, 16, ⊙; 25, 0, 90°. Pitch, 24. Generating point, o, 10, 16, 0. Point c, x, y, 4. Point d, x, y, 10.

6003. Helix is left handed. Axis, line A, 25, 0, 90°; 25, 16, ⊙. Pitch, 24. Generating point, o, 25, 0, 2. Point c, 25, 12, 30. Point d, 25, 24, 2.

6004. Helix is right handed. Axis, line A, 25, 0, 90°; 25, 16, ⊙. Pitch, 24. Generating point, o, 39, 0, 16. Point c, x, 10, z. Point d, x, 16, z.

Problem 61. Draw the projections of a right helicoid for one complete turn of the directing helix. The pitch should be divided into 12 equal parts. Find the other projection of the given point on the surface, and pass a tangent plane at this point. Each (2).

6101. Directing helix is left handed. Axis, line A, 28, 17, ⊙; 28, 0, 90°. Pitch, 24. Generating point, o, 13, 17, 0. Point c, 34, y, 8.

6102. Directing helix is left handed. Axis, line A, 25, 16, ⊙; 25, 0, 90°. Pitch, 24. Generating point, o, 39, 16, 0. Point c, 30, y, 4.

6103. Directing helix is right handed. Axis, line A, 29, 17, ⊙; 29, 0, 90°. Pitch, 24. Generating point, o, 14, 17, 0. Point c, 21, y, 14.

6104. Directing helix is right handed. Axis, line A, 20, 15, ⊙; 20, 0, 90°. Pitch 24. Generating point, o, 8, 15, 0. Point c, 25, 20, z.

6105. Directing helix is left handed. Axis, line A, 25, 17, ⊙; 25, 0, 90°. Pitch, 30. Generating point, o, 11, 17, 0. Point c, 25, 10, z.

6106. Directing helix is left handed. Axis, line A, 25, 18, ⊙; 25, 0, 90°. Pitch, 30. Generating point, o, 25, 4, 0. Point c, 34, 18, z.

Problem 62. Draw the projections of a right helicoid for one complete turn of the directing helix. The pitch should be divided into 12 equal parts. Find the intersection of the surface with the given plane. Each (2).

6201. Directing helix is left handed. Axis, line A, 28, 16, ⊙; 28, 0, 90°. Pitch, 24. Generating point, o, 14, 16, 0. Plane Q, 24, 120°, 45°.

6202. Directing helix is right handed. Axis, line A, 30, 16, ⊙; 30, 0, 90°. Pitch, 30. Generating point, o, 44, 16, 0. Plane Q, 32, 60°, 135°.

6203. Directing helix is left handed. Axis, line A, 25, 16, ⊙; 25, 0, 90°. Pitch, 24. Generating point, o, 25, 2, 0. Plane Q, 28, 60°, 120°.

6204. Directing helix is left handed. Axis, line A, 22, 16, ⊙; 22, 0, 90°. Pitch, 24. Generating point, o, 35, 16, 0. Plane Q, 42, 120°, 135°.

6205. Directing helix is left handed. Axis, line A, 22, 16, ⊙; 22, 0, 90°. Pitch, 24. Generating point, o, 22, 30, 0. Plane Q, 25, 135°, 45°.

6206. Directing helix is right handed. Axis, line A, 25, 15, ⊙; 25, 0, 90°. Pitch, 24. Generating point, o, 25, 28, 0. Plane Q, ∞, 21, 22.

Problem 63. Draw the projections of one nappe of an oblique helicoid for one complete turn (or less if the space will not permit) of the directing helix. The pitch should be divided into 12 equal parts. Find the other projection of the given point on the surface, and pass a tangent plane at this point. Each (2).

6301. Directing helix is right handed. Axis, line A, 16, 16, ⊙; 16, 0, 90°. Pitch, 18. Generating line, B, 3, 16, 0°; 3, 0, 30°. Point c, 20, 9, z.

6302. Directing helix is left handed. Axis, line A, 28, 14, ⊙; 28, 0, 90°. Pitch, 24. Generating line, B, 15, 14, 0°; 15, 0, 45°. Point c, 24, 7, z.

6303. Directing helix is right handed. Axis, line A, 25, 15, ⊙; 25, 0, 90°. Pitch, 18. Generating line, B, 39, 15, 0°; 39, 0, 150°. Point c, 21, 8, z.

6304. Directing helix is left handed. Axis, line A, 30, 16, ⊙; 30, 0, 90°. Pitch, 24. Generating line, ob; o, 30, 30, 0; b, 30, 16, 14. Point c, 22, 16, z.

6305. Directing helix is left handed. Axis, line A, 20, 15, ⊙; 20, 0, 90°. Pitch, 18. Generating line, B, 34, 15, 0°; 34, 0, 135°. Point c, 24, y, 9.

6306. Directing helix is right handed. Axis, line A, 18, 16, ⊙; 18, 0, 90°. Pitch, 18. Generating line, B, 32, 16, 0°; 32, 0, 135°. Point c, 18, 9, z.

Problem 64. Draw the projections of one nappe of an oblique helicoid. The pitch of the directing helix should be divided into 12 equal parts. Find the intersection of the surface with the given plane, so far as the space will permit, extending the surface in either direction from the given position of the generating line. Draw a line tangent to the intersection at some point. Each (2).

6401. Directing helix is right handed. Axis, line A, 19, 16, ⊙; 19, 0, 90°. Pitch, 18. Generating line, B, 31, 16, 0°; 31, 0, 120°. Plane Q, 30, 45°, 150°.

6402. Directing helix is right handed. Axis, line A, 25, 13, ⊙; 25, 0, 90°. Pitch, 18. Generating line, B, 13, 13, 0°; 13, 0, 60°. Plane Q, 18, 135°, 30°.

6403. Directing helix is left handed. Axis, line A, 19, 17, ⊙; 19, 0, 90°. Pitch, 24. Generating line, B, 5, 17, 0°; 5, 0, 45°. Plane Q, 47, 120°, 150°.

6404. Directing helix is left handed. Axis, line A, 30, 16, ⊙; 30, 0, 90°. Pitch, 24. Generating line, B, 16, 16, 0°; 16, 0, 45°. Plane Q, 14, 105°, 30°.

6405. Directing helix is left handed. Axis, line A, 21, 14, ⊙; 21, 0, 90°. Pitch, 24. Generating line, ob; o, 21, 2, 0; b, 21, 14, 12. Plane Q, 17, 135°, 30°.

6406. Directing helix is right handed. Axis, line A, 25, 12, ⊙; 25, 0, 90°. Pitch, 24. Generating line, B, 13, 12, 0°; 13, 0, 45°. Plane Q, ∞, 28, 28.

Problem 65. Draw the projections of a right conoid, showing both nappes. Find the other projections of the given points in the surface, and pass a tangent plane at each point. Each (2).

6501. Line of striction, A, 25, 0, 90°; 25, 20, ⊙. Base, a circle in H; diameter 28, center at e, 25, 16, 0. Points c and d, 17, y, 5.

6502. Line of striction, A, 25, 0, 90°; 25, 18, ⊙. Base, a circle in H; diameter 26, center at e, 25, 16, 0. Points c and d, 30, 6, z.

6503. Line of striction, A, 30, 16, 45°; 30, 19, 0°. Base, a circle in H; diameter 28, center at e, 30, 16, 0. Points c and d, 26, 6, z.

6504. Line of striction, A, 25, 16, 30°; 25, 16, 0°. Base, a circle in H; diameter 28, center at e, 25, 16, 0. Points c and d, 18, 7, z.

6505. Line of striction, A, 22, 0, 90°; 22, 16, ⊙. Base, an ellipse in H; center at e, 22, 16, 0; axes, 28 and 20, major axis parallel to line of striction. Points c and d, 28, y, 5.

6506. Line of striction, A, 22, 16, 150°; 22, 20, 0°. Base, an ellipse in H; center at e, 22, 16, 0; axes, 28 and 20, major axis parallel to line of striction. Point c, 30, 8, z, where z is less than 20.

Problem 66. Draw the projections of a right conoid, showing both nappes. Take at least 12 elements.

In the following examples, find the intersection of the surface with the given plane. Draw a line tangent to the intersection at some point, except points on elements which have a single tangent plane along their entire length (§ 244). Each (2).

6601. Line of striction, A, 20, 0, 90°; 20, 20, ⊙. Base, a circle in H; diameter 28, center at e, 20, 16, 0. Plane Q, 48, 120°, 150°.

6602. Line of striction, A, 27, 0, 90°; 27, 20, ⊙. Base, a circle in H; diameter 28, center at e, 27, 15, 0. Plane Q, 17, 135°, 15°.

6603. Line of striction, A, 23, 16, 45°; 23, 18, 0°. Base, a circle in H; diameter 28, center at e, 23, 16, 0. Plane Q, 27, 60°, 120°.

6604. Line of striction, A, 12, 0, 90°; 12, 18, ⊙. Base, a circle in H; diameter 22, center at e, 12, 12, 0. Plane Q, ∞, 21, 30.

In the following examples, the given point lies between the base and line of striction. Pass a plane tangent to the surface at this point. Find the intersection of this plane with the surface. Each (2).

6605. Line of striction, A, 25, 16, 45°; 25, 18, 0°. Base, a circle in H; diameter 24, center at e, 25, 16, 0. Point c, 20, 8, z.

6606. Line of striction, A, 28, 16, 45°; 28, 18, 0°. Base, an ellipse in H; center at e, 28, 16, 0; axes, 28 and 20, minor axis parallel to line of striction. Point c, 25, 9, z.

6607. Line of striction, A, 28, 13, 45°; 28, 16, 0°. Base, an ellipse in H; center at e, 28, 13, 0; axes, 28 and 22, major axis parallel to line of striction. Point c, 20, 7, z.

APPLIED PROBLEMS

A 1. (8) First quadrant. A pipe runs through a factory building from a point m, 1, 4, 16, to point n, 20, 16, $-$ 6, in the floor below. What is its length? At what point in the floor (H) must the hole for it be made? At a point 6 from the floor hole, measured along the pipe, is a support from the floor. How far from the floor and the wall (V) is this point of support?

A 2. (3) First quadrant. A smokestack, 50 feet high, diameter 3 feet, rises from point a, 17, 14, 0, on top of a flat roof (H). Five guy wires are attached to the stack 30 feet above the roof, and are anchored at points on the roof, respectively b, 5, 4, 0; c, 27, 3, 0; d, 29, 19, 0; e, 21, 27, 0; and f, 3, 21, 0. Draw the stack and guys, and measure the true length of each guy. Scale 1 unit = 1 foot.

A 3. (8) First quadrant. A cogwheel railroad runs from the summit of a hill, a, 3, 2, 16, to a point in the valley, b, 20, 12, 4. The road is straight and the grade uniform. Find the length of one rail and the grade. (The grade is the angle with the horizontal.) Scale 1 unit = 400 feet.

A 4. (8) Two boys rigged a string telephone between their rooms. The ends of the line are at points a, 3, 3, 16 and b, 3, 14, 4. Find the length of string required. Scale 2 units = 1 foot.

A 5. (8) Third quadrant. The upper surface of a vein of coal is found to be perpendicular to a shaft bored in the direction cd; c, 24, 3, 1; d, 9, 14, 16. The ore is located at point o, 16, y, z, on cd. Draw the traces to the plane of the ore.

A 6. (6) First quadrant. The vertical walls of a building are planes M, ∞, 2, ∞, and N; HN, 26, 7, 60°; VN, not drawn, 90°. An inclined platform is to be built in the (obtuse) corner, running from

the bottom of a door opening *B*, 26, 7, 60°; 26, 8, 0°, to the ground (*H*). The line *C*, 26, 7, 150°; 26, 8, 150°, is the outer edge of the platform; its inner edge is against the wall *M*. Complete the two views of the platform.

A 7. (3) First quadrant. The plan of a shed with a sloping roof is given by points *a*, 3, 17, 0; *b*, 23, 27, 0; *c*, 30, 13, 0; *d*, 10, 3, 0. The front (long) wall of the shed is 12 high; the back wall is 20 high. In the center of the roof is an opening for a skylight; *e*, 13, 17, *z*; *f*, 17, 19, *z*; *j*, 20, 13, *z*; *k*, 16, 11, *z*. Draw the elevation of the shed and skylight.

A 8. (10) First quadrant. The top of the mast of a derrick is at point *a*, 2, 15, 16. Good anchorage for a guy rope can be had at *b*, 18, 6, 0. There is an old shed in the way. Plan of shed, *cdef*; *c*, 8, 9, 0; *d*, 16, 13, 0; *e*, 18, 9, 0; *f*, 10, 5, 0. Height of front (long) edge of wall, 4; of back wall, 7. Find the points where holes must be cut in roof and wall to let the rope through.

A 9. (8) A ray of light, represented by line *L*, 8, 9, 150°; 8, 13, 135°, emanates from point *b*, 8, 9, 13. A mirror is located at *cdef*; *c*, 11, 4, 12; *d*, 11, 4, 4; *e*, 16, 17, 4; *f*, 16, 7, 12. Find the point in which the ray of light strikes the mirror, and the true angle between the ray and the plane of the mirror.

A 10. (8) First quadrant. A flagpole *F*, 19, 5, 150°; 19, 0, 150°, is fastened to a sloping roof *abc*; *a*, 7, 4, 0; *b*, 22, 4, 9; *c*, 17, 14, 0. Find the point of attachment, and the angle between the pole and the roof.

A 11. (6) First quadrant. A telephone line running over a hill is shown by line *A*, 11, 4, 135°; 11, 6, 30°, and line *B*, 11, 4, 0°; 11, 6, 150°. It is possible to make connection from point *c*, 13, 10, 2, to either *A* or *B*. Find the shortest route and show it in both views.

A 12. (6) A telephone wire is represented by line *A*, 13, 0, 60°; 19, 0, 150°. A high-tension power line is represented by *B*, 30, 0, 150°; 4, 0, 45°. If the safe distance between these lines is 2 feet, is the given location safe, and by what margin? Scale 4 units = 1 foot.

A 13. (3) The roof of a square tower, side 18, is a regular pyramid, altitude 24. Draw plan and elevation of the pyramidal roof. Find the true slope of its sides, and the true angle between two adjacent sides.

A 14. (2) A hopper has the form of the inverted frustum of a regular square pyramid. Side of base, 12; of top, 24; height, 20. Draw plan and elevation. Find the true size of one side, and the true angle between any two adjacent inclined sides.

A 15. (2) Portal of Skew Bridge. Scale $\frac{1}{2}''$ = 1 ft. See Fig. 382, left side. Given a portion of the end post (slanting) and top chord (horizontal) of the two trusses of a through bridge. The distance in the clear between the trusses has been much reduced, to save space.

Draw the plan and elevation of the outside member of the sway bracing, namely, a rectangular strut, 6″ × 18″, connecting the trusses, and butted against the inner faces of the end posts. The long edges of the strut are parallel to the line *cd*; the top edge lies in the plane of the top of the trusses. The strut is centered on the inner faces of the end posts, its front (wide) face parallel to the plane which contains the lines *A* and *B*. Draw also views of the strut showing in true size (*a*) its front (wide) face, and (*b*) its top (narrow) face.

Draw a bent bracing plate under each end of the strut. This plate is of thin sheet metal, and its thickness may be neglected in the drawing.

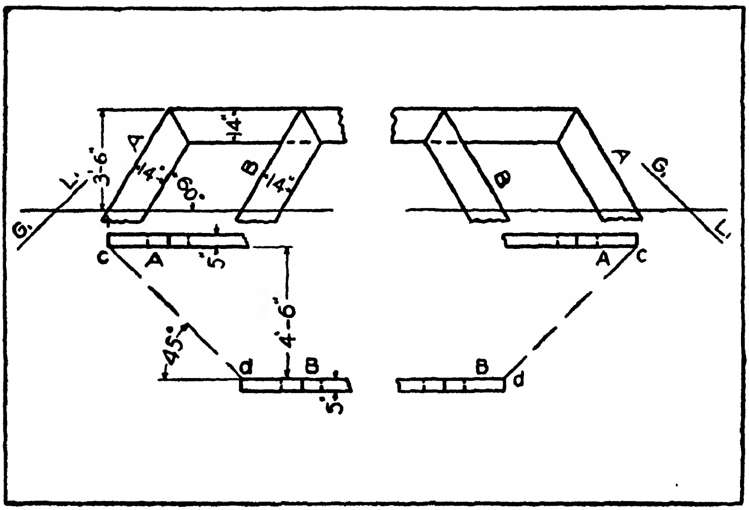

Fig. 382.

The leaf which extends under the strut is the width of the strut; the leaf which extends down the side of the end post is of a width to correspond. Make each leaf a parallelogram, all long edges 20″. Find the pattern for each bent plate. Find also the angle for bending each plate, that is, the true angle between its faces.

Suggestion. A vertical plane, perpendicular to *cd*, cuts the strut in its true cross section. Hence the strut may be placed by constructing an auxiliary projection, the secondary ground line being taken as suggested in Fig. 382.

A **16. (2) Portal of Skew Bridge. Scale ½″ = 1 ft. See Fig. 382, right side. The strut is not centered on the inner faces of the end posts,**

but its front (wide) face is in the plane of the lines *A* and *B*. Otherwise this example is the same as the preceding.

A 17. (2) Hip Roof. Scale $\frac{3}{4}''$ = 1 ft. See Fig. 383, left side. Given the plan of the corner of a building. The outside edges of the two wall plates are *HQ* and *HR*. The side *Q* of the roof has a slope of 4 (horizontal) to 3 (vertical); the side *R* slopes 1 to 1. Find the (plan of the) line of intersection of the two roof planes.

A hip rafter, 6″ × 8″, is centered on this line of intersection. The jack rafters are 4″ × 8″, and their center lines meet in pairs on that of the hip rafter. The upper ends of the jack rafters are butted against

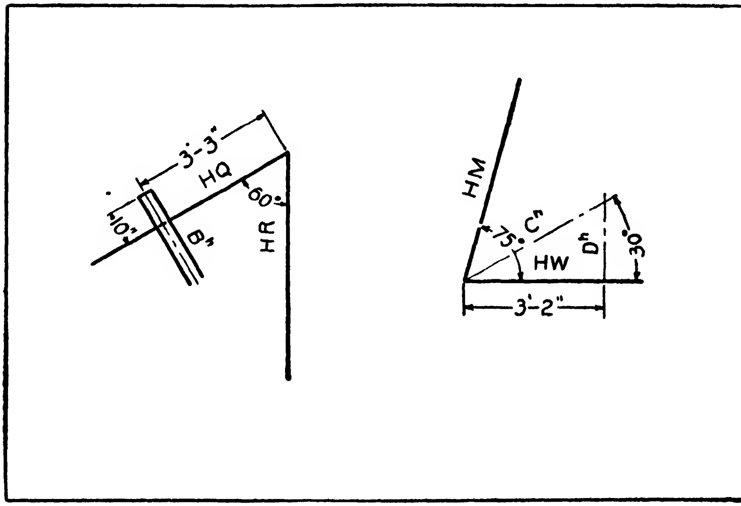

Fig. 383.

the side of the hip rafters. Draw the plan view of the hip rafter and a pair of jack rafters, of which *B* is one. Cut off the outer ends of the rafters by vertical planes which meet on the center line of the hip rafter.

Imagine the planes *Q* and *R* raised 6″ vertically. The tops of the jack rafters lie in this new position of the planes, and the top of the hip rafter is beveled on both sides of its center line to coincide with the planes. Draw a true-size side view of each rafter, showing in each a rectangular notch at the outer corner of the wall. Draw also a true-size top view of each rafter.

A 18. (2) Hip Roof. Scale $\frac{3}{4}''$ = 1 ft. See Fig. 383, right side. Given the plan of the corner of a building. The hip, or line of inter-

section of the roof planes, is C, which has a slope of 3 (horizontal) to 2 (vertical). The center line of a jack rafter is D; its mate on the other roof intersects C at the same point. Find the following angles, using the center lines only; the angle of each rafter with the horizontal (3 angles); the angle of each rafter with the plumb line or vertical (3 angles); the angle between each jack and the hip (2 angles). Show where corresponding angles appear in the side and top views of the rafters in the preceding example. Find also the angle between the roof planes M and W. What would this angle be used for?

A 19. (2) Hip Roof. Scale $\frac{3}{4}''$ = 1 ft. See Fig. 384, left side. The directions for this example are the same as for *A* 17.

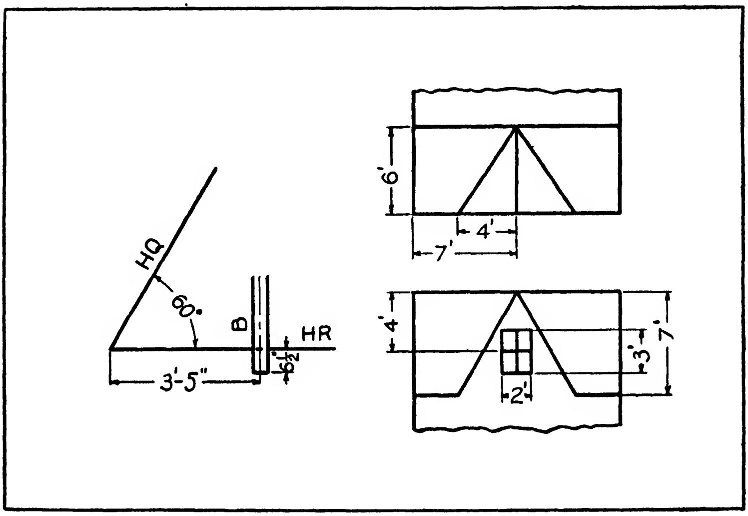

FIG. 384.

A 20. (2) Valley Roof. Scale $\frac{1}{4}''$ = 1 ft. See Fig. 384, right side. Given a pitch roof with a dormer, the intersection of the main and dormer roofs forming a valley. Using center lines only, find these angles: the slope of the main roof; the slope of the dormer roof; the slope of the valley; the angle between a jack rafter in the main roof and the valley rafter; the angle between a jack rafter in the dormer roof and the valley rafter; the angle between the main and dormer roofs.

A 21. (1) Conical Elbow and Vertical Pipe. Scale $\frac{3}{4}''$ = 1 ft. See Fig. 385. Complete the plan view. Make a half development of the elbow and pipe.

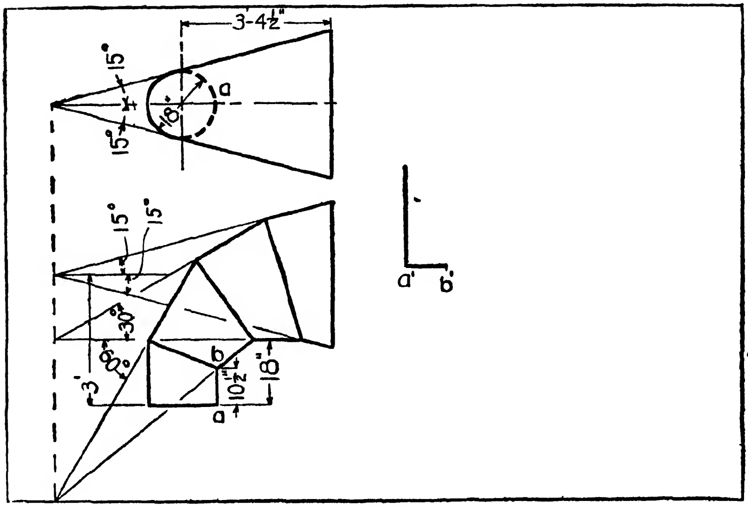

FIG. 385

A 22. (1) Register Box and Supply Pipe. Scale 3″ = 1 ft. See Fig. 386. Develop the elbow piece *A*, opening it on the shortest element. Develop the connecting piece *B*, opening it on the shortest element.

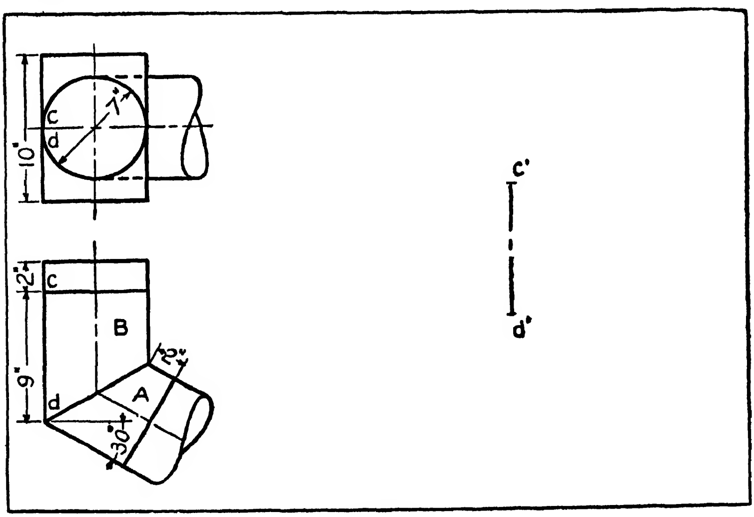

FIG. 386.

A 23. (1) Ash Chute. Scale $1\frac{1}{2}''$ = 1 ft. See Fig. 387. The connecting piece between the bottom of the square opening and the side of the vertical pipe is composed partly of planes (triangles) and partly of conical surfaces. Make a half development of this connection. Develop one half of the vertical pipe, showing the opening in its side.

A 24. (1) Offset Connection for Furnace Pipes. Scale 3″ = 1 ft. See Fig. 388. The offset connection is composed partly of planes (triangles) and partly of cylindrical surfaces. Design a suitable connection, and obtain its complete development. Develop the cylindrical portions by triangulation (Prob. 42).

A 25. (1) Ventilator Pipe. Scale 3″ = 1 ft. See Fig. 389. A pipe of circular section is converted to one of rectangular section, and

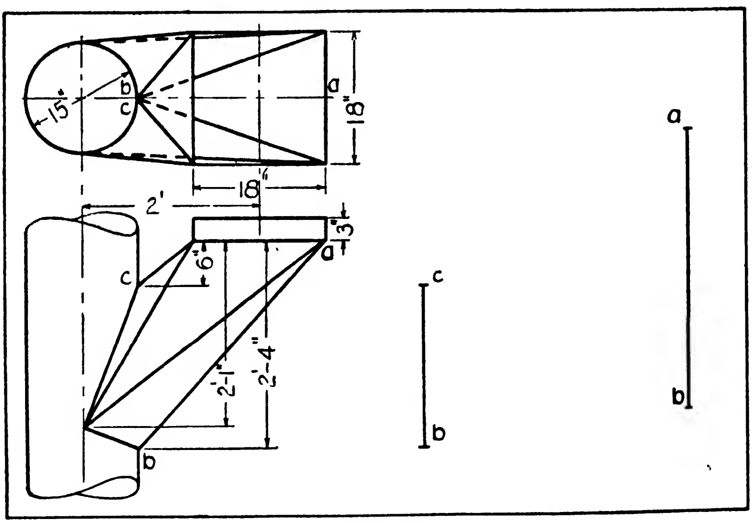

Fig. 387.

at the same time is offset. The transition piece is composed partly of planes (triangles) and partly of conical surfaces. Design a suitable transition piece, and find its development, making the joint along *ec*. Develop also the upper section of the circular pipe, opening it along the shortest element.

A 26. (1) Y-Branch Piping. Scale $1\frac{1}{2}''$ = 1 ft. See Fig. 390. A circular pipe is divided into two circular pipes by means of a connecting piece. Each half of the connection, as *E*, is cut from a cone,

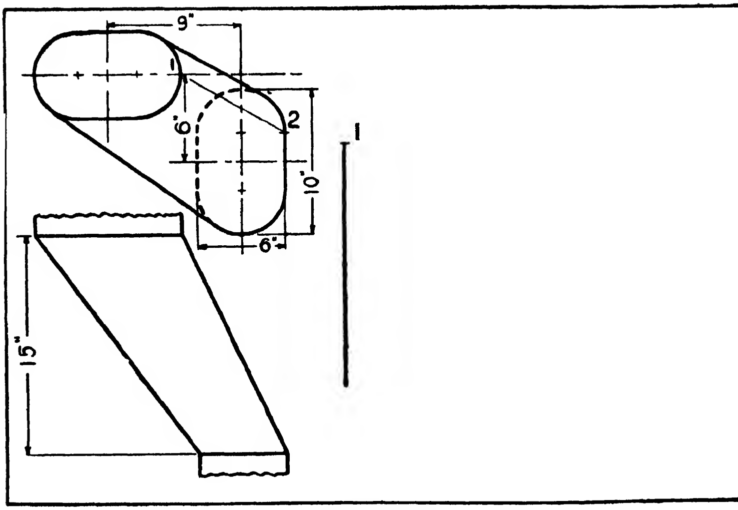

Fig. 388.

oab, which has circular sections in the two directions indicated. Develop the piece *E*. Find the true size of the section *cd*. Make a half development of the piece *F*.

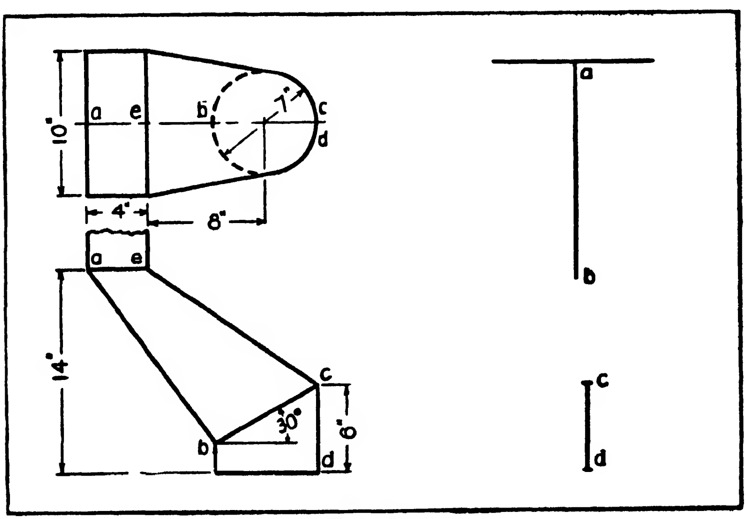

Fig. 389.

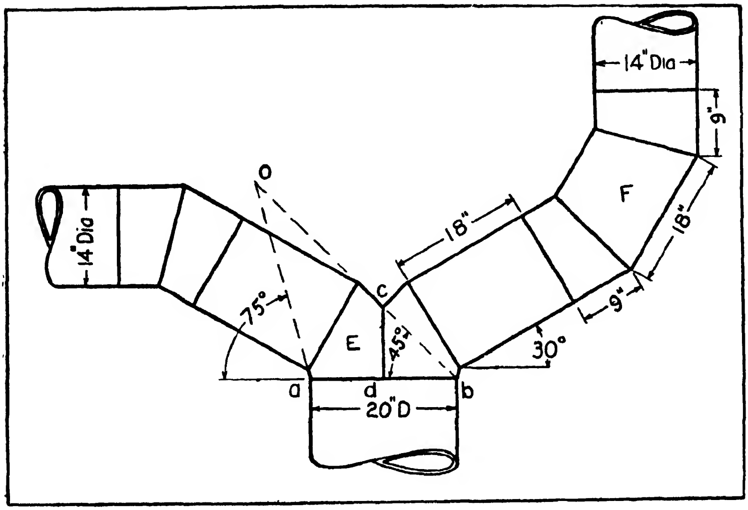

FIG. 390.

A 27. (1) Screw Propeller. Scale 1″ = 1 ft. See Fig. 391. The propeller is right handed and has four blades. The hub is spherical. The surface of each blade is a portion of a right helicoid, pitch 11′ – 0″. No thickness to the blades need be shown. The given curve is the

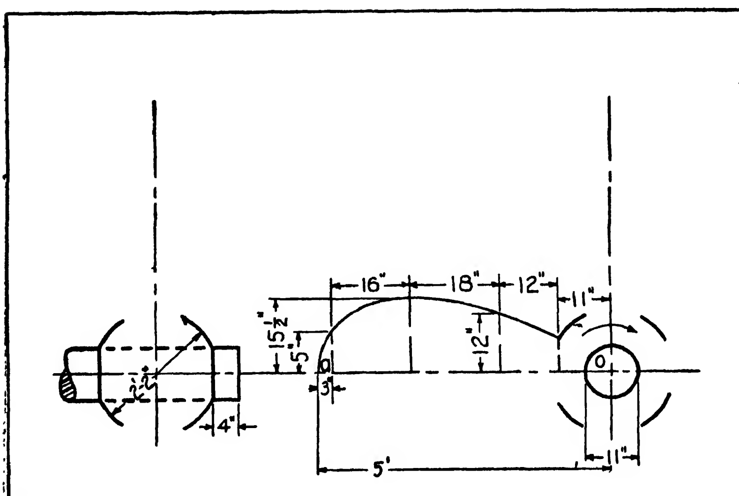

FIG. 391.

half development of one blade, showing the true width of the blade at various points. Find the front and side elevations of one horizontal blade and one vertical blade.

SUGGESTION. Assume that a line tangent to any helix of the surface at the point where it crosses the element *oa* lies in the surface of the blade. Determine this tangent by developing the helix for a length measured along its axis, equal to the pitch divided by 2π. Take $\pi = \frac{22}{7}$.

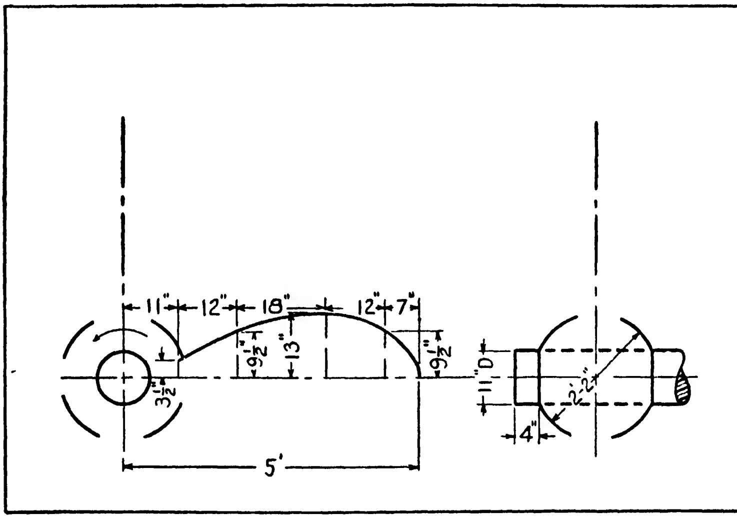

FIG. 392.

A 28. (1) Screw Propeller. Scale 1" = 1 ft. See Fig. 392. The propeller is left handed and has four blades. The pitch is 11′ – 0″. The hub is spherical. No thickness to the blades need be considered The given curve is the half elevation of one blade. Draw the complete front and side elevations of one horizontal and one vertical blade. Take $\pi = \frac{22}{7}$.